VORONOVSKAYA TYPE ASYMPTOTIC EXPANSIONS FOR SMOOTH SYMMETRIZED AND PERTURBED HYPERBOLIC TANGENT ORDINARY AND FRACTIONAL NEURAL NETWORK APPROXIMATIONS OVER INFINITE DOMAINS

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ABSTRACT. In this article we research the univariate smooth approximation ordinary and fractional under differentiation of functions. The approximators here are neural network operators activated by the symmetrized and perturbed hyperbolic tangent function. All domains here are of the whole real line. The neural network operators here are of quasi-interpolation type: the basic ones, the Kantorovich type ones, and of the quadrature type. We give Voronovskaya type asymptotic expansions.

1. Introduction

The author in [1] and [2], see 'Chapters 2-5' was the first to establish quantitative neural network approximation to continuous functions with rates by very specific neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by using the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" activation functions are assumed to be of compact support.

Again the author inspired by [14], continued his studies on neural network approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent types which resulted into [5] - [9], by treating both the univariate and multivariate cases. The author also treated the corresponding fractional cases [10], [11].

The author here presents symmetrized and perturbed hyperbolic tangent activated neural network Voronovskaya asymptotic expansions for differentiated functions from $\mathbb{R}$ into $\mathbb{R}$.

For a detailed study in neural networks you may choose [18]-[20].

A multilayer feed-forward neural network can be defines as follows (with $m \in \mathbb{N}$ hidden layers):

Let $x \in \mathbb{R}^s$; $s \in \mathbb{N}$, where $x = (x_1, ..., x_s)$; $\alpha_j, c_j \in \mathbb{R}^s$; $b_j \in \mathbb{R}$, with $0 \leq j \leq n$, $n \in \mathbb{N}$.

Here $\langle \alpha_j \cdot x \rangle$ is the inner product, thus $\sigma(\langle \alpha_j \cdot x \rangle + b_j) \in \mathbb{R}$; and $N_n(x) \in \mathbb{R}^s$, by $c_j \in \mathbb{R}^s$, as it is coming from $N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle \alpha_j \cdot x \rangle + b_j)$.

We define:

$$N_n^{(2)}(x) = \sum_{j=0}^{n} c_j \sigma(\langle \alpha_j \cdot N_n(x) \rangle + b_j) =$$

$$\sum_{j=0}^{n} c_j \sigma\left(\left\langle \alpha_j \cdot \left(\sum_{j=0}^{n} c_j \sigma(\langle \alpha_j \cdot x \rangle + b_j)\right)\right\rangle + b_j\right).$$

Furthermore, we can define

$$N_n^{(3)}(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle \alpha_j \cdot N_n^{(2)}(x) \rangle + b_j\right).$$

And, in general we define:

$$N_n^{(m)}(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle \alpha_j \cdot N_n^{(m-1)}(x) \rangle + b_j\right), \quad \text{for } m \in \mathbb{N}.$$

2. Basics

Initially we follow [12], pp. 455-460.

Our perturbed hyperbolic tangent activation function here to be used is

$$(1) \qquad g_{q,\lambda}(x) := \frac{e^{\lambda x} - q e^{-\lambda x}}{e^{\lambda x} + q e^{-\lambda x}}, \quad \lambda, q > 0, \ x \in \mathbb{R}.$$

Here λ is the parameter and q is the deformation coefficient.

For more details read Chapter 18 of [12]: "q-deformed and λ-Parametrized Hyperbolic Tangent based Banach space Valued Ordinary and Fractional Neural Network Approximation".

'The Chapters 17 and 18' of [12] motivate our current work.

The proposed "symmetrization method" aims to use half data feed to our neural networks.

We will employ the following density function

$$(2) \qquad M_{q,\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0,$$

$\forall x \in \mathbb{R}$; $q, \lambda > 0$.

We have that

$$(3) \qquad M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \quad \forall\, x \in \mathbb{R}; \ q, \lambda > 0,$$

and

$$(4) \qquad M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x), \quad \forall\, x \in \mathbb{R}; \ q, \lambda > 0.$$

Adding (3) and (4) we obtain

$$(5) \qquad M_{q,\lambda}(-x) + M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x),$$

a key to this work.

So that

$$(6) \qquad \Phi(x) := \frac{M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x)}{2}$$

is an even function, symmetric with respect to the y-axis.

By (18.18) of [12], we have

$$(7) \qquad \begin{aligned} &M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \\ &\text{and} \\ &M_{\frac{1}{q},\lambda}\left(-\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \quad \lambda > 0. \end{aligned}$$

sharing the same maximum at symmetric points.

By Theorem 18.1, p. 458 of [12], we have that

$$(8) \qquad \begin{aligned} &\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \quad \forall\, x \in \mathbb{R}, \ \lambda, q > 0, \\ &\text{and} \\ &\sum_{i=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x-i) = 1, \quad \forall\, x \in \mathbb{R}, \ \lambda, q > 0. \end{aligned}$$

Consequently, we derive that

$$(9) \qquad \sum_{i=-\infty}^{\infty} \Phi(x-i) = 1, \quad \forall\, x \in \mathbb{R}.$$

By Theorem 18.2, p. 459 of [12], we have that

$$(10) \qquad \begin{aligned} &\int_{-\infty}^{\infty} M_{q,\lambda}(x)\, dx = 1, \\ &\text{and} \\ &\int_{-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x)\, dx = 1, \end{aligned}$$

so that

(11) $$\int_{-\infty}^{\infty} \Phi\left(x\right) dx = 1,$$

therefore Φ is a density function.

By Theorem 18.3, p. 459 of [4], we have:

Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q, \lambda > 0$. Then

(12)

$$\sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\alpha}}}^{\infty} M_{q,\lambda}\left(nx - k\right) < 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = T e^{-2\lambda n^{(1-\alpha)}},$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Similarly, we get that

(13) $$\sum_{k=-\infty}^{\infty} M_{\frac{1}{q},\lambda}\left(nx - k\right) < T e^{-2\lambda n^{(1-\alpha)}}.$$

Consequently we obtain that

(14)

$$\sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\alpha}}}^{\infty} \Phi\left(nx - k\right) < T e^{-2\lambda n^{(1-\alpha)}},$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

An essential property follows:

Theorem 2.1. ([13]) *It holds*

(15) $$\Phi\left(x\right) < \left(q + \frac{1}{q}\right) \lambda e^{-2\lambda(x-1)}, \quad \forall\, x \geq 1.$$

We need,

Definition 2.2. In this article we study the smooth Voronovskaya type asymptotic expansion properties of the following interpolation neural network operators acting on $f \in C\left(\mathbb{R}\right)$ (continuous functions):

(i) the basic ones

(16) $$B_n\left(f, x\right) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi\left(nx - k\right), \quad \forall\, x \in \mathbb{R},\, n \in \mathbb{N},$$

(ii) the Kantorovich type operators

(17) $$C_n\left(f, x\right) := \sum_{k=-\infty}^{\infty} \left(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f\left(t\right) dt\right) \Phi\left(nx - k\right), \quad \forall\, x \in \mathbb{R},\, n \in \mathbb{N},$$

(iii) let $\theta \in \mathbb{N}$, $w_r \geq 0$, $\sum_{r=0}^{\theta} w_r = 1$, $k \in \mathbb{Z}$, and

$$(18) \qquad \delta_{nk}(f) := \sum_{r=0}^{\theta} w_r f \left(\frac{k}{n} + \frac{r}{n\theta} \right),$$

we consider also the quadrature type operators

$$(19) \qquad D_n(f, x) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) \Phi(nx - k), \quad \forall\, x \in \mathbb{R}, \, n \in \mathbb{N}.$$

Here $C_B(\mathbb{R})$ are the continuous and bounded functions on $\mathbb{R}$.

3. Main Results

We start with the basic operators.

Theorem 3.1. *Here* $0 < \beta < 1$, $n \in \mathbb{N}$ *large enough,* $x \in \mathbb{R}$, $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, *with* $f^{(N)} \in C_B(\mathbb{R})$, $0 < \varepsilon \leq N$. *Then*

1)

$$(20) \qquad B_n(f, x) - f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} B_n\left((\cdot - x)^j \right)(x) + o\left(\frac{1}{n^{\beta(N-\varepsilon)}} \right),$$

the last (20) implies

$$(21) \qquad n^{\beta(N-\varepsilon)} \left[B_n(f, x) - f(x) - \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} B_n\left((\cdot - x)^j \right)(x) \right] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

2) when $f^{(j)}(x) = 0$, $j = 1, ..., N$, *we derive that*

$$(22) \qquad n^{\beta(N-\varepsilon)} \left[B_n(f, x) - f(x) \right] \to 0, \quad \text{as } n \to \infty, \, 0 < \varepsilon \leq N.$$

Of interest is the case $\beta = \frac{1}{2}$.

Proof. By Taylor's theorem we have $(x \in \mathbb{R})$

$$(23) \quad f\left(\frac{k}{n} \right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x \right)^j + \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t \right)^{N-1}}{(N-1)!} dt.$$

It follows

$$f\left(\frac{k}{n} \right) \Phi(nx - k) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \Phi(nx - k) \left(\frac{k}{n} - x \right)^j +$$

$$\Phi(nx - k) \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t \right)^{N-1}}{(N-1)!} dt.$$

Hence

$$(24) \qquad B_n(f,x) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} B_n\left((\cdot - x)^j\right)(x) +$$

$$\sum_{k=-\infty}^{\infty} \Phi(nx-k) \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt.$$

Call

$$(25) \qquad R := \sum_{k=-\infty}^{\infty} \Phi(nx-k) \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt.$$

(I) Let $\left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}$. Then

i) case $\frac{k}{n} \geq x$:

$$(26) \quad |R| \leq \sum_{\substack{k=-\infty \\ : \left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}}}^{\infty} \Phi(nx-k) \int_x^{\frac{k}{n}} \left|f^{(N)}(t) - f^{(N)}(x)\right| \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt$$

$$\leq 2\left\|f^{(N)}\right\|_\infty \sum_{\substack{k=-\infty \\ : \left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}}}^{\infty} \Phi(nx-k) \frac{\left(\frac{k}{n} - x\right)^N}{N!}$$

$$\leq 2\left\|f^{(N)}\right\|_\infty \frac{1}{N!n^{\beta N}}.$$

We found that

$$(27) \qquad |R|\big|_{\left|\frac{k}{n}-x\right| < \frac{1}{n^\beta}} \leq 2\left\|f^{(N)}\right\|_\infty \frac{1}{N!n^{\beta N}}.$$

(ii) case $\frac{k}{n} < x$: then

$$(28) \quad |R| \leq \sum_{\substack{k=-\infty \\ : \left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}}}^{\infty} \Phi(nx-k) \left|\int_{\frac{k}{n}}^x \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt\right|$$

$$\leq \sum_{\substack{k=-\infty \\ : \left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}}}^{\infty} \Phi(nx-k) \int_{\frac{k}{n}}^x \left|f^{(N)}(x) - f^{(N)}(t)\right| \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt$$

$$\leq 2\left\|f^{(N)}\right\|_\infty \sum_{\substack{k=-\infty \\ : \left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}}}^{\infty} \Phi(nx-k) \frac{\left(x - \frac{k}{n}\right)^N}{N!}$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty \frac{1}{N! n^{\beta N}}.$$

Consequently, we have proved that

$$(29) \qquad |R| \big|_{\left| \frac{k}{n} - x \right| < \frac{1}{n^\beta}} \leq \frac{2 \left\| f^{(N)} \right\|_\infty}{N! n^{\beta N}}.$$

Next, we see $\left(\frac{k}{n} \geq x \right)$

$$|R| \leq \sum_{\substack{k = -\infty \\ : \left| \frac{k}{n} - x \right| \geq \frac{1}{n^\beta}}}^{\infty} \Phi(nx - k) \left| \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t \right)^{N-1}}{(N-1)!} dt \right|$$

$$\leq \sum_{\substack{k = -\infty \\ : \left| \frac{k}{n} - x \right| \geq \frac{1}{n^\beta}}}^{\infty} \Phi(nx - k) \int_x^{\frac{k}{n}} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left(\frac{k}{n} - t \right)^{N-1}}{(N-1)!} dt$$

$$(30) \qquad \leq 2 \left\| f^{(N)} \right\|_\infty \sum_{\substack{k = -\infty \\ : \left| \frac{k}{n} - x \right| \geq \frac{1}{n^\beta}}}^{\infty} \Phi(nx - k) \frac{\left(\frac{k}{n} - x \right)^N}{N!}$$

$$= \frac{2 \left\| f^{(N)} \right\|_\infty}{n^N N!} \sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\beta}}}^{\infty} \Phi(nx - k) (k - nx)^N.$$

In case $\frac{k}{n} < x$, we get

$$|R| \leq \sum_{\substack{k = -\infty \\ : \left| \frac{k}{n} - x \right| \geq \frac{1}{n^\beta}}}^{\infty} \Phi(nx - k) \left| \int_{\frac{k}{n}}^x \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(t - \frac{k}{n} \right)^{N-1}}{(N-1)!} dt \right|$$

$$\leq \sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\beta}}}^{\infty} \Phi(nx - k) \int_{\frac{k}{n}}^x \left| f^{(N)}(x) - f^{(N)}(t) \right| \frac{\left(t - \frac{k}{n} \right)^{N-1}}{(N-1)!} dt$$

$$(31) \qquad \leq 2 \left\| f^{(N)} \right\|_\infty \sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\beta}}}^{\infty} \Phi(nx - k) \frac{\left(x - \frac{k}{n} \right)^N}{N!}$$

$$= \frac{2\left\|f^{(N)}\right\|_\infty}{n^N N!} \sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}}^{\infty} \Phi\left(nx - k\right)\left(nx - k\right)^N.$$

Consequently, it holds

$$(32) \qquad |R|\,|_{\left|\frac{k}{n}-x\right| \geq \frac{1}{n^\beta}} \leq \frac{2\left\|f^{(N)}\right\|_\infty}{n^N N!} \sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}}^{\infty} \Phi\left(|nx - k|\right)|nx - k|^N.$$

Next, we treat

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}}^{\infty} \Phi\left(|nx - k|\right)|nx - k|^N \overset{(15)}{\leq}$$

$$\left(\mu := \left(q + \frac{1}{q}\right)\lambda\right)$$

$$\mu \sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}}^{\infty} e^{-2\lambda(|nx-k|-1)}|nx - k|^N =$$

$$(33) \qquad \mu e^{2\lambda} \sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}}^{\infty} e^{-2\lambda|nx-k|}|nx - k|^N =: (*).$$

Notice that (set $\overline{\mu} := 2\lambda$)

$$(34) \qquad e^{\frac{\overline{\mu}|nx-k|}{2}} = \sum_{\lambda=0}^{\infty} \frac{\left(\frac{\overline{\mu}|nx-k|}{2}\right)^\lambda}{\lambda!} \geq \left(\frac{\overline{\mu}\,|nx - k|}{2}\right)^N \frac{1}{N!}.$$

Therefore we have

$$\left(\frac{\overline{\mu}\,|nx - k|}{2}\right)^N \leq N! e^{\frac{\overline{\mu}|nx-k|}{2}}, \quad \text{or}$$

$$(35) \qquad \left(\overline{\mu}\,|nx - k|\right)^N \leq 2^N N! e^{\frac{\overline{\mu}|nx-k|}{2}}, \quad \text{or}$$

$$|nx - k|^N \leq \frac{2^N}{\overline{\mu}^N} N! e^{\frac{\overline{\mu}|nx-k|}{2}}.$$

Hence it holds

$$(*) \leq \frac{\mu}{\overline{\mu}^N} e^{\overline{\mu}} 2^N N! \sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}}^{\infty} e^{-\overline{\mu}|nx-k|} e^{\frac{\overline{\mu}|nx-k|}{2}} =$$

$$\frac{\mu}{\overline{\mu}^N} e^{\overline{\mu}} 2^N N! \left(\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}}^{\infty} e^{-\frac{\overline{\mu}}{2}|nx-k|} \right) \leq$$

$$\frac{\mu}{\overline{\mu}^N} 2 e^{\overline{\mu}} 2^N N! \left(\int_{n^{1-\beta}-1}^{\infty} e^{-\frac{\overline{\mu}}{2}x} dx \right) =$$

(36)
$$\frac{\mu}{\overline{\mu}^N} \frac{2}{\overline{\mu}} 2 e^{\overline{\mu}} 2^N N! \left(\int_{n^{1-\beta}-1}^{\infty} e^{-\frac{\overline{\mu}}{2}x} d\frac{\overline{\mu}x}{2} \right) \overset{(y:=\frac{\overline{\mu}x}{2})}{=}$$

$$\frac{\mu}{\overline{\mu}^{N+1}} 2^{N+2} e^{\overline{\mu}} N! \left(\int_{n^{1-\beta}-1}^{\infty} e^{-y} dy \right) = \frac{\mu}{\overline{\mu}^{N+1}} 2^{N+2} e^{\overline{\mu}} N! \left(e^{-y} \big|_{\infty}^{n^{1-\beta}-1} \right) =$$

$$\frac{\mu}{\overline{\mu}^{N+1}} 2^{N+2} e^{\overline{\mu}} N! e^{-\frac{\overline{\mu}}{2}\left(n^{1-\beta}-1\right)} = \frac{\left(q+\frac{1}{q}\right)}{\lambda^N} 2 e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

Thus we have

(37)
$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}}^{\infty} \Phi\left(|nx-k|\right) |nx-k|^N \leq \frac{\left(q+\frac{1}{q}\right)}{\lambda^N} 2 e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

And, it is

(38)
$$|R| \big|_{\left|\frac{k}{n}-x\right| \geq \frac{1}{n^\beta}} \leq \left(\frac{2\left\|f^{(N)}\right\|_\infty}{N! n^N} \right) \frac{\left(q+\frac{1}{q}\right)}{\lambda^N} 2 e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}$$

$$= \frac{4\left\|f^{(N)}\right\|_\infty}{n^N \lambda^N} \left(q+\frac{1}{q}\right) e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

We proved that

(39)
$$|R| \big|_{\left|\frac{k}{n}-x\right| \geq \frac{1}{n^\beta}} \leq \frac{4\left\|f^{(N)}\right\|_\infty}{n^N \lambda^N} \left(q+\frac{1}{q}\right) e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

Finally, we derive that

(40)
$$|R| \leq |R| \big|_{\left|\frac{k}{n}-x\right| < \frac{1}{n^\beta}} + |R| \big|_{\left|\frac{k}{n}-x\right| \geq \frac{1}{n^\beta}} \leq$$

$$\frac{2\left\|f^{(N)}\right\|_\infty}{n^{\beta N} N!} + \frac{4\left\|f^{(N)}\right\|_\infty}{n^N \lambda^N} \left(q+\frac{1}{q}\right) e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

So, for large enough $n \in \mathbb{N}$ we get

$$(41) \qquad |R| \le \frac{4\left\|f^{(N)}\right\|_\infty}{n^{\beta N} N!}.$$

Hence it holds

$$(42) \qquad |R| = O\left(\frac{1}{n^{\beta N}}\right),$$

and

$$(43) \qquad |R| = o(1).$$

And, letting $0 < \varepsilon \le N$, we derive

$$(44) \qquad \frac{|R|}{\left(\frac{1}{n^{\beta(N-\varepsilon)}}\right)} \le \frac{4\left\|f^{(N)}\right\|_\infty}{N!}\left(\frac{1}{n^{\beta\varepsilon}}\right) \to 0, \quad \text{as } n \to \infty.$$

I.e.

$$(45) \qquad |R| = o\left(\frac{1}{n^{\beta(N-\varepsilon)}}\right),$$

proving the claim. $\qquad\qquad\square$

The Kantorovich operators follow.

Theorem 3.2. *Let $0 < \beta < 1$, $n \in \mathbb{N}$ large enough, $x \in \mathbb{R}$, $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, with $f^{(N)} \in C_B(\mathbb{R})$, $0 < \varepsilon \le N$. Then*

1)

$$(46) \qquad C_n(f,x) - f(x) = \sum_{j=1}^N \frac{f^{(j)}(x)}{j!} C_n\left((\cdot - x)^j\right)(x) + o\left(\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^{N-\varepsilon}\right),$$

the last (46) implies

$$(47) \qquad \frac{1}{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^{N-\varepsilon}}\left[C_n(f,x) - f(x) - \sum_{j=1}^N \frac{f^{(j)}(x)}{j!} C_n\left((\cdot - x)^j\right)(x)\right] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \le N$.

2) when $f^{(j)}(x) = 0$, $j = 1, ..., N$, we derive that

$$(48) \qquad \frac{1}{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^{N-\varepsilon}}\left[C_n(f,x) - f(x)\right] \to 0, \quad \text{as } n \to \infty, \ 0 < \varepsilon \le N.$$

Of interest is the case $\beta = \frac{1}{2}$

Proof. One can write

$$(49) \qquad C_n(f,x) = \sum_{k=-\infty}^\infty \left(n \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) G(nx - k).$$

Let now $f \in C^N(\mathbb{R})$ with $f^{(N)} \in C_B(\mathbb{R})$, $N \in \mathbb{N}$.

We have that

$$(50) \qquad f\left(t+\frac{k}{n}\right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!}\left(t+\frac{k}{n}-x\right)^{j}+$$

$$\int_{x}^{t+\frac{k}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!}ds,$$

and

$$n\int_{0}^{\frac{1}{n}}f\left(t+\frac{k}{n}\right)dt = \sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}n\int_{0}^{\frac{1}{n}}\left(t+\frac{k}{n}-x\right)^{j}dt+$$

$$(51) \qquad n\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{k}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt.$$

Hence

$$C_{n}(f,x) = \sum_{k=-\infty}^{\infty}\left(n\int_{0}^{\frac{1}{n}}f\left(t+\frac{k}{n}\right)dt\right)\Phi(nx-k) =$$

$$(52) \qquad \sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}C_{n}\left((\cdot-x)^{j}\right)(x)+$$

$$\sum_{k=-\infty}^{\infty}\Phi(nx-k)\left(n\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{k}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right).$$

Therefore we can write

$$(53) \qquad C_{n}(f,x)-f(x)-\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}C_{n}\left((\cdot-x)^{j}\right)(x) = R,$$

where
(54)
$$R := \sum_{k=-\infty}^{\infty}\Phi(nx-k)\left(n\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{k}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right).$$

Call

$$(55) \qquad \lambda(k) := n\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{k}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt,$$

where $k\in\mathbb{Z}$.

I) Let $\left|\frac{k}{n}-x\right| < \frac{1}{n^{\beta}}$ $(0<\beta<1)$.

i) if $t+\frac{k}{n}\geq x$, then

$$|\lambda(k)|\leq n\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{k}{n}}\left|f^{(N)}(s)-f^{(N)}(x)\right|\frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt$$

$$(56) \qquad \leq 2\left\|f^{(N)}\right\|_{\infty}n\int_{0}^{\frac{1}{n}}\frac{\left(t+\frac{k}{n}-x\right)^{N}}{N!}dt$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty n \int_0^{\frac{1}{n}} \frac{\left(|t| + \left| \frac{k}{n} - x \right| \right)^N}{N!} dt$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\frac{1}{n} + \frac{1}{n^\beta} \right)^N}{N!}.$$

(ii) if $t + \frac{k}{n} < x$, then

$$|\lambda(k)| \leq n \int_0^{\frac{1}{n}} \left| \int_{t+\frac{k}{n}}^x \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{k}{n} - s \right)^{N-1}}{(N-1)!} ds \right| dt$$

$$\leq n \int_0^{\frac{1}{n}} \left(\int_{t+\frac{k}{n}}^x \left| f^{(N)}(x) - f^{(N)}(s) \right| \frac{\left(s - \left(t + \frac{k}{n} \right) \right)^{N-1}}{(N-1)!} ds \right) dt$$

(57)
$$\leq 2 \left\| f^{(N)} \right\|_\infty n \int_0^{\frac{1}{n}} \frac{\left(x - t - \frac{k}{n} \right)^N}{N!} dt$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty n \int_0^{\frac{1}{n}} \frac{\left(\left| x - \frac{k}{n} \right| + |t| \right)^N}{N!} dt$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\frac{1}{n^\beta} + \frac{1}{n} \right)^N}{N!}.$$

Therefore, when $\left| \frac{k}{n} - x \right| < \frac{1}{n^\beta}$, then

(58)
$$|\lambda(k)| \leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\frac{1}{n^\beta} + \frac{1}{n} \right)^N}{N!}.$$

Clearly now it holds

(59)
$$|R| \big|_{\left| \frac{k}{n} - x \right| < \frac{1}{n^\beta}} \leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\frac{1}{n} + \frac{1}{n^\beta} \right)^N}{N!}.$$

II) Let $\left| \frac{k}{n} - x \right| \geq \frac{1}{n^\beta}$.

i) if $t + \frac{k}{n} \geq x$, then

(60)
$$|\lambda(k)| \leq 2 \left\| f^{(N)} \right\|_\infty n \int_0^{\frac{1}{n}} \frac{\left(t + \frac{k}{n} - x \right)^N}{N!} dt$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty n \int_0^{\frac{1}{n}} \frac{\left(\left| \frac{k}{n} - x \right| + |t| \right)^N}{N!} dt$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\left| \frac{k}{n} - x \right| + \frac{1}{n} \right)^N}{N!}.$$

ii) if $t + \frac{k}{n} < x$, then

$$|\lambda(k)| \leq n \int_0^{\frac{1}{n}} \left(\int_{t+\frac{k}{n}}^x \left| f^{(N)}(x) - f^{(N)}(s) \right| \frac{\left(s - \left(t + \frac{k}{n} \right) \right)^{N-1}}{(N-1)!} ds \right) dt$$

(61)
$$\leq 2 \left\| f^{(N)} \right\|_\infty n \int_0^{\frac{1}{n}} \frac{\left(x - t - \frac{k}{n} \right)^N}{N!} dt$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty n \int_0^{\frac{1}{n}} \frac{\left(\left| \frac{k}{n} - x \right| + |t| \right)^N}{N!} dt \leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\left| \frac{k}{n} - x \right| + \frac{1}{n} \right)^N}{N!}.$$

Hence when $\left| \frac{k}{n} - x \right| \geq \frac{1}{n^\beta}$, then

(62)
$$|\lambda (k)| \leq \frac{2 \left\| f^{(N)} \right\|_\infty}{n^N N!} (|nx - k| + 1)^N$$

$$\leq \frac{2^N \left\| f^{(N)} \right\|_\infty}{n^N N!} \left(1 + |nx - k|^N \right), \quad \forall \ k \in \mathbb{Z}.$$

Clearly, then

$$|R| |_{\left| \frac{k}{n} - x \right| \geq \frac{1}{n^\beta}} \leq \left(\sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\beta}}}^{\infty} \Phi (nx - k) |\lambda (k)| \right) \leq$$

$$\frac{2^N \left\| f^{(N)} \right\|_\infty}{n^N N!} \left(\sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\beta}}}^{\infty} \Phi (|nx - k|) \left(1 + |nx - k|^N \right) \right) =$$

$$\frac{2^N \left\| f^{(N)} \right\|_\infty}{n^N N!} \left[\sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\beta}}}^{\infty} \Phi (|nx - k|) + \right.$$

(63)
$$\left. \left(\sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\beta}}}^{\infty} \Phi (|nx - k|) |nx - k|^N \right) \right] \leq$$

(by (14), (37))

$$\frac{2^N \left\| f^{(N)} \right\|_\infty}{n^N N!} \left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q + \frac{1}{q} \right)}{\lambda^N} 2 e^{2\lambda} N! e^{-\lambda \left(n^{1-\beta} - 1 \right)} \right].$$

We have found that

(64)
$$|R| |_{\left| \frac{k}{n} - x \right| \geq \frac{1}{n^\beta}} \leq \frac{2^N \left\| f^{(N)} \right\|_\infty}{n^N N!} \left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q + \frac{1}{q} \right)}{\lambda^N} 2 e^{2\lambda} N! e^{-\lambda \left(n^{1-\beta} - 1 \right)} \right].$$

Therefore, it holds

$$|R| \le |R| \, |_{\left|\frac{k}{n}-x\right|<\frac{1}{n^\beta}} + |R| \, |_{\left|\frac{k}{n}-x\right|\ge\frac{1}{n^\beta}} \le$$

(65)
$$2 \left\| f^{(N)} \right\|_\infty \frac{\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^N}{N!} +$$

$$\frac{2^N \left\| f^{(N)} \right\|_\infty}{n^N N!} \left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q+\frac{1}{q}\right)}{\lambda^N} 2e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)} \right].$$

For large enough $n \in \mathbb{N}$ we have that

(66)
$$|R| \le \frac{4 \left\| f^{(N)} \right\|_\infty}{N!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^N.$$

Hence it holds

(67)
$$|R| = O\left(\left(\frac{1}{n} + \frac{1}{n^\beta} \right)^N \right),$$

and

(68)
$$|R| = o\left(1\right).$$

And, letting $0 < \varepsilon \le N$, we derive

(69)
$$\frac{|R|}{\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^{N-\varepsilon}} \le \frac{4 \left\| f^{(N)} \right\|_\infty}{N!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^\varepsilon \to 0, \quad \text{as } n \to \infty.$$

I.e.

(70)
$$|R| = o\left(\left(\frac{1}{n} + \frac{1}{n^\beta} \right)^{N-\varepsilon} \right),$$

proving the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We continue with the quadrature operators.

Theorem 3.3. *Let* $0 < \beta < 1$, $n \in \mathbb{N}$ *large enough*, $x \in \mathbb{R}$, $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, *with* $f^{(N)} \in C_B(\mathbb{R})$, $0 < \varepsilon \le N$. *Then*

1)

(71)
$$D_n(f,x) - f(x) = \sum_{j=1}^N \frac{f^{(j)}(x)}{j!} D_n\left((\cdot - x)^j \right)(x) + o\left(\left(\frac{1}{n} + \frac{1}{n^\beta} \right)^{N-\varepsilon} \right),$$

the last (71) implies

(72)
$$\frac{1}{\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^{N-\varepsilon}} \left[D_n(f,x) - f(x) - \sum_{j=1}^N \frac{f^{(j)}(x)}{j!} D_n\left((\cdot - x)^j \right)(x) \right] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \le N$.

2) when $f^{(j)}(x) = 0$, $j = 1, ..., N$, we derive that

(73)
$$\frac{1}{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^{N-\varepsilon}} [D_n(f, x) - f(x)] \to 0, \quad as\ n \to \infty,\ 0 < \varepsilon \le N.$$

Of interest is the case $\beta = \frac{1}{2}$

Proof. We have that

(74)
$$f\left(\frac{k}{n} + \frac{i}{nr}\right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n} + \frac{i}{nr} - x\right)^j +$$

$$\int_x^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!} dt,$$

and

(75)
$$\sum_{i=1}^{r} w_i f\left(\frac{k}{n} + \frac{i}{nr}\right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \sum_{i=1}^{r} w_i \left(\frac{k}{n} + \frac{i}{nr} - x\right)^j +$$

$$\sum_{i=1}^{r} w_i \int_x^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!} dt.$$

Furthermore it holds

(76)
$$D_n(f, x) - f(x) - \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} D_n\left((\cdot - x)^j\right)(x) = R(x),$$

where
(77)
$$R(x) = \sum_{k=-\infty}^{\infty} \Phi(nx - k) \left(\sum_{i=1}^{r} w_i \int_x^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!} dt\right).$$

Call

(78)
$$\gamma(k) := \sum_{i=1}^{r} w_i \int_x^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!} dt.$$

I) Let $\left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}$.

i) if $\frac{k}{n} + \frac{i}{nr} \ge x$, then

$$|\gamma(k)| \le \sum_{i=1}^{r} w_i 2 \left\|f^{(N)}\right\|_\infty \frac{\left(\left|\frac{k}{n} - x\right| + \frac{i}{nr}\right)^N}{N!}$$

(79)
$$\le 2\left\|f^{(N)}\right\|_\infty \frac{\left(\frac{1}{n^\beta} + \frac{1}{n}\right)^N}{N!}.$$

ii) if $\frac{k}{n} + \frac{i}{nr} < x$, then

$$|\gamma(k)| \le \sum_{i=1}^{r} w_i \left|\int_x^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!} dt\right|$$

$$\leq \sum_{i=1}^{r} w_i \int_{\frac{k}{n}+\frac{i}{nr}}^{x} \left| \left(f^{(N)}(x) - f^{(N)}(t) \right) \right| \frac{\left(t - \left(\frac{k}{n}+\frac{i}{nr} \right) \right)^{N-1}}{(N-1)!} dt$$

$$\leq \sum_{i=1}^{r} w_i 2 \left\| f^{(N)} \right\|_\infty \frac{\left(x - \left(\frac{k}{n}+\frac{i}{nr} \right) \right)^N}{N!}$$

$$(80) \qquad\qquad \leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\frac{1}{n^\beta}+\frac{1}{n} \right)^N}{N!}.$$

Therefore, when $\left| \frac{k}{n} - x \right| < \frac{1}{n^\beta}$, then

$$(81) \qquad\qquad |\gamma(k)| \leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\frac{1}{n^\beta}+\frac{1}{n} \right)^N}{N!}.$$

Clearly now it holds

$$(82) \qquad\qquad |R| \big|_{\left| \frac{k}{n}-x \right| < \frac{1}{n^\beta}} \leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\frac{1}{n}+\frac{1}{n^\beta} \right)^N}{N!}.$$

II) Let $\left| \frac{k}{n} - x \right| \geq \frac{1}{n^\beta}$.

i) if $\frac{k}{n} + \frac{i}{nr} \geq x$, then

$$|\gamma(k)| \leq \sum_{i=1}^{r} w_i 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\frac{k}{n}+\frac{i}{nr}-x \right)^N}{N!}$$

$$(83) \qquad\qquad \leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\left| \frac{k}{n}-x \right|+\frac{1}{n} \right)^N}{N!}.$$

ii) if $\frac{k}{n} + \frac{i}{nr} < x$, then

$$|\gamma(k)| \leq \sum_{i=1}^{r} w_i \int_{\frac{k}{n}+\frac{i}{nr}}^{x} \left| f^{(N)}(x) - f^{(N)}(t) \right| \frac{\left(t - \left(\frac{k}{n}+\frac{i}{nr} \right) \right)^{N-1}}{(N-1)!} dt$$

$$(84) \qquad\qquad \leq 2 \left\| f^{(N)} \right\|_\infty \sum_{i=1}^{r} w_i \frac{\left(\left| x - \frac{k}{n} \right|+\frac{i}{nr} \right)^N}{N!}$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\left| x - \frac{k}{n} \right|+\frac{1}{n} \right)^N}{N!}.$$

So, in general we obtain

$$|\gamma(k)| \leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\left| x - \frac{k}{n} \right|+\frac{1}{n} \right)^N}{N!}$$

$$= \frac{2 \left\| f^{(N)} \right\|_\infty}{n^N} \frac{\left(|nx - k|+1 \right)^N}{N!} \leq$$

$$(85) \qquad\qquad \frac{2^N \left\| f^{(N)} \right\|_\infty}{n^N N!} \left(1 + |nx - k|^N \right), \quad \forall\, k \in \mathbb{Z}.$$

Clearly, then

$$
(86) \qquad |R(x)|\,\big|_{\left|\frac{k}{n}-x\right|\geq\frac{1}{n^\beta}} \leq \left(\sum_{\substack{k=-\infty \\ :\,|nx-k|\geq n^{1-\beta}}}^{\infty} \Phi(nx-k)\,|\gamma(k)|\right) \leq
$$

$$
\frac{2^N\,\|f^{(N)}\|_\infty}{n^N N!}\left(\sum_{\substack{k=-\infty \\ :\,|nx-k|\geq n^{1-\beta}}}^{\infty}\Phi(|nx-k|)\left(1+|nx-k|^N\right)\right)
$$

$$
\leq \text{(as earlier)}
$$

$$
(87) \qquad \leq \frac{2^N\,\|f^{(N)}\|_\infty}{n^N N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^N}2e^{2\lambda}N!\,e^{-\lambda\left(n^{1-\beta}-1\right)}\right].
$$

Therefore, it holds

$$
|R(x)|\leq|R(x)|\,\big|_{\left|\frac{k}{n}-x\right|<\frac{1}{n^\beta}}+|R(x)|\,\big|_{\left|\frac{k}{n}-x\right|\geq\frac{1}{n^\beta}}\leq
$$

$$
(88) \qquad 2^N\,\|f^{(N)}\|_\infty\frac{\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^N}{N!}+
$$

$$
\frac{2^N\,\|f^{(N)}\|_\infty}{n^N N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^N}2e^{2\lambda}N!\,e^{-\lambda\left(n^{1-\beta}-1\right)}\right].
$$

For large enough $n\in\mathbb{N}$ we have that

$$
(89) \qquad |R(x)|\leq\frac{4\,\|f^{(N)}\|_\infty}{N!}\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^N.
$$

Hence it holds

$$
(90) \qquad |R(x)|=O\left(\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^N\right),
$$

and

$$
(91) \qquad |R(x)|=o(1).
$$

And, letting $0<\varepsilon\leq N$, we derive

$$
(92) \qquad \frac{|R(x)|}{\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^{N-\varepsilon}}\leq\frac{4\,\|f^{(N)}\|_\infty}{N!}\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^\varepsilon\to 0,\ \text{ as }n\to\infty.
$$

I.e.

$$(93) \qquad |R(x)| = o\left(\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^{N-\varepsilon}\right),$$

proving the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We need,

Definition 3.4. A function $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous over $\mathbb{R}$, iff $f|_{[a,b]}$ is absolutely continuous, for every $[a, b] \subset \mathbb{R}$. We write $f \in AC^n(\mathbb{R})$, iff $f^{(n-1)} \in AC(R)$ (absolutely continuous functions over $\mathbb{R}$), $n \in \mathbb{N}$.

Definition 3.5. Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n(\mathbb{R})$. The left Caputo fractional derivative ([17], [21], [15], pp. 49-52) of the function f is defined by

$$(94) \qquad D_{*a}^\nu f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt,$$

$\forall\, x \in [a, \infty)$, $a \in \mathbb{R}$, where Γ is the gamma function.

Notice $D_{*a}^\nu f \in L_1([a,b])$ and $D_{*a}^\nu f$ exists a.e. on $[a,b]$, $\forall\, [a,b] \subset \mathbb{R}$.

We set $D_{*a}^0 f(x) = f(x)$, $\forall\, x \in [a, \infty)$.

We need

Lemma 3.6. *(see also [4]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}(\mathbb{R})$ and $f^{(n)} \in L_\infty(\mathbb{R})$. Then $D_{*a}^\nu f(a) = 0$ for any $a \in \mathbb{R}$.*

Definition 3.7. (see also [3, 16, 17]) Let $f \in AC^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by

$$(95) \qquad D_{b-}^\alpha f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz,$$

$\forall\, x \in (-\infty, b]$, $b \in \mathbb{R}$. We set $D_{b-}^0 f(x) = f(x)$.

Notice $D_{b-}^\alpha f \in L_1([a,b])$ and $D_{b-}^\alpha f$ exists a.e. on $[a,b]$, $\forall\, [a,b] \subset \mathbb{R}$.

Lemma 3.8. *(see also [4]) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^\alpha f(b) = 0$, for any $b \in \mathbb{R}$.*

Convention 3.9. We assume that

$$(96) \qquad \begin{aligned} &D_{*x_0}^\alpha f(x) = 0, \text{ for } x < x_0, \\ &\text{and} \\ &D_{x_0-}^\alpha f(x) = 0, \text{ for } x > x_0. \end{aligned}$$

We mention

Proposition 3.10. (see also [4]) Let $f \in C^n(\mathbb{R})$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^\nu f(x)$ is continuous in $x \in [a, \infty)$, $a \in \mathbb{R}$.

Also we have

Proposition 3.11. (see also [4]) Let $f \in C^m (\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D^\alpha_{b-} f(x)$ is continuous in $x \in (-\infty, b]$, $b \in \mathbb{R}$.

We further mention

Proposition 3.12. (see also [4]) Let $f \in C^{m-1} (\mathbb{R})$, $f^{(m)} \in L_\infty (\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and let $x, x_0 \in \mathbb{R} : x \geq x_0$. Then $D^\alpha_{*x_0} f(x)$ is continuous in x_0.

Proposition 3.13. (see also [4]) Let $f \in C^{m-1} (\mathbb{R})$, $f^{(m)} \in L_\infty (\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and let $x, x_0 \in \mathbb{R} : x \leq x_0$. Then $D^\alpha_{x_0-} f(x)$ is continuous in x_0.

Proposition 3.14. (see also [4]) Let $f \in C^m (\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$; $x, x_0 \in \mathbb{R}$. Then $D^\alpha_{*x_0} f(x)$, $D^\alpha_{x_0-} f(x)$ are jointly continuous functions in (x, x_0) from $R^2 \to \mathbb{R}$.

Fractional results follow.

Next we treat the fractional case of Voronovskaya type asymptotic expansion for the basic operators.

Theorem 3.15. *Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in AC^N (\mathbb{R})$, $f^{(N)} \in L_\infty (\mathbb{R})$, $0 < \beta < 1$, $x \in \mathbb{R}$, $n \in \mathbb{N}$ large enough. Assume that both $\|D^\alpha_{*x} f\|_{\infty, [x, \infty)}$, $\|D^\alpha_{x-} f\|_{\infty, (-\infty, x]}$ are finite, $0 < \varepsilon \leq \alpha$. Then*

1)

$$(97) \qquad B_n(f)(x) - f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} B_n \left((\cdot - x)^j \right)(x) + o \left(\frac{1}{n^{\beta(N-\varepsilon)}} \right),$$

the last (97) implies

$$(98) \qquad n^{\beta(N-\varepsilon)} \left[B_n(f)(x) - f(x) - \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} B_n \left((\cdot - x)^j \right)(x) \right] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

2) when $N = 1$, or $f^{(j)}(x) = 0$, $j = 1, ..., N - 1$, we get that

$$n^{\beta(N-\varepsilon)} [B_n(f)(x) - f(x)] \to 0, \quad as \ n \to \infty, \ 0 < \varepsilon \leq N.$$

Of interest is the case $\beta = \frac{1}{2}$.

Proof. Let $x \in \mathbb{R}$. We have that $D^\alpha_{x-} f(x) = D^\alpha_{*x} f(x) = 0$.

From [15], p. 54, we get the left Caputo fractional Taylor's formula that

$$(99) \qquad f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j +$$

$$\frac{1}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} (D^\alpha_{*x} f(J) - D^\alpha_{*x} f(x)) \, dJ,$$

for all $x \leq \frac{k}{n} < \infty$.

Also from [3], using the right Caputo fractional Taylor's formula we get

$$(100) \qquad f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j +$$

$$\frac{1}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha}f(J) - D_{x-}^{\alpha}f(x)\right) dJ,$$

for all $-\infty < \frac{k}{n} \leq x$.

Hence it holds

$$(101) \qquad f\left(\frac{k}{n}\right) \Phi(nx-k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \Phi(nx-k) \left(\frac{k}{n} - x\right)^j +$$

$$\frac{\Phi(nx-k)}{\Gamma(\alpha)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left(D_{*x}^{\alpha}f(J) - D_{*x}^{\alpha}f(x)\right) dJ,$$

for all $x \leq \frac{k}{n} < \infty$, iff $\lceil nx \rceil \leq k < \infty$,

and

$$(102) \qquad f\left(\frac{k}{n}\right) \Phi(nx-k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \Phi(nx-k) \left(\frac{k}{n} - x\right)^j +$$

$$\frac{\Phi(nx-k)}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha}f(J) - D_{x-}^{\alpha}f(x)\right) dJ,$$

for all $-\infty < \frac{k}{n} \leq x$, iff $-\infty < k \leq \lfloor nx \rfloor$.

We have that $\lceil nx \rceil \leq \lfloor nx \rfloor + 1$.

Therefore it holds

$$(103) \qquad \sum_{k=\lfloor nx \rfloor + 1}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lfloor nx \rfloor + 1}^{\infty} \Phi(nx-k) \left(\frac{k}{n} - x\right)^j +$$

$$\frac{\sum_{k=\lfloor nx \rfloor + 1}^{\infty} \Phi(nx-k)}{\Gamma(\alpha)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left(D_{*x}^{\alpha}f(J) - D_{*x}^{\alpha}f(x)\right) dJ,$$

and

$$(104) \qquad \sum_{k=-\infty}^{\lfloor nx \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=-\infty}^{\lfloor nx \rfloor} \Phi(nx-k) \left(\frac{k}{n} - x\right)^j +$$

$$\frac{\sum_{k=-\infty}^{\lfloor nx \rfloor} \Phi(nx-k)}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha}f(J) - D_{x-}^{\alpha}f(x)\right) dJ.$$

Adding the last two equalities (103), (104) we obtain

$$(105) \qquad B_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} B_n\left((\cdot - x)^j\right)(x) = R_n(x),$$

where

$$(106) \qquad R_n(x) := R_{1n}(x) + R_{2n}(x),$$

with

$$(107) \qquad R_{1n}(x) := \frac{\sum\limits_{k=\lfloor nx \rfloor + 1}^{\infty} \Phi(nx - k)}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha - 1} (D_{*x}^\alpha f(J) - D_{*x}^\alpha f(x)) \, dJ,$$

and

$$(108) \qquad R_{2n}(x) := \frac{\sum\limits_{k=-\infty}^{\lfloor nx \rfloor} \Phi(nx - k)}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha - 1} \left(D_{x-}^\alpha f(J) - D_{x-}^\alpha f(x)\right) \, dJ.$$

Furthermore, let

$$(109) \qquad \delta_{1n}(x) := \frac{1}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha - 1} (D_{*x}^\alpha f(J) - D_{*x}^\alpha f(x)) \, dJ,$$

for $k = \lfloor nx \rfloor + 1, ..., \infty$,

and

$$(110) \qquad \delta_{2n}(x) := \frac{1}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha - 1} \left(D_{x-}^\alpha f(J) - D_{x-}^\alpha f(x)\right) \, dJ,$$

for $k = -\infty, ..., \lfloor nx \rfloor$.

Let $\left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}$, we derive that

$$(111) \qquad |\delta_{1n}(x)| \leq \|D_{*x}^\alpha f\|_{\infty, [x, \infty)} \frac{1}{n^{\alpha\beta}\Gamma(\alpha + 1)},$$

$k = \lfloor nx \rfloor + 1, ..., \infty$,

and

$$(112) \qquad |\delta_{2n}(x)| \leq \|D_{x-}^\alpha f\|_{\infty, (-\infty, x]} \frac{1}{n^{\alpha\beta}\Gamma(\alpha + 1)},$$

$k = -\infty, ..., \lfloor nx \rfloor$.

Also we obtain that

$$(113) \qquad |\delta_{1n}(x)| \leq \|D_{*x}^\alpha f\|_{\infty, [x, \infty)} \frac{\left(\frac{k}{n} - x\right)^\alpha}{\Gamma(\alpha + 1)}, \quad k = \lfloor nx \rfloor + 1, ..., \infty,$$

and

$$(114) \qquad |\delta_{2n}(x)| \leq \|D_{x-}^\alpha f\|_{\infty, (-\infty, x]} \frac{\left(x - \frac{k}{n}\right)^\alpha}{\Gamma(\alpha + 1)}, \quad k = -\infty, ..., \lfloor nx \rfloor.$$

Therefore, it holds

(115)
$$|R_{1n}(x)|\,|_{\left|x-\frac{k}{n}\right|<\frac{1}{n^{\beta}}} \leq \frac{\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}}{n^{\alpha\beta}\Gamma(\alpha+1)},$$

$k = \lfloor nx \rfloor + 1, ..., \infty,$

and

(116)
$$|R_{2n}(x)|\,|_{\left|x-\frac{k}{n}\right|<\frac{1}{n^{\beta}}} \leq \frac{\|D_{x-}^{\alpha}f\|_{\infty,(-\infty,x]}}{n^{\alpha\beta}\Gamma(\alpha+1)},$$

$k = -\infty, ..., \lfloor nx \rfloor.$

Next, we estimate

$$|R_{1n}(x)|\,|_{\left|x-\frac{k}{n}\right|\geq\frac{1}{n^{\beta}}} \leq$$

$$\frac{\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}}{\Gamma(\alpha+1)} \sum_{\substack{k=\lfloor nx \rfloor+1 \\ :\left|x-\frac{k}{n}\right|\geq\frac{1}{n^{\beta}}}}^{\infty} \Phi(nx-k)\left(\frac{k}{n}-x\right)^{\alpha} \leq$$

(by $N = \lceil \alpha \rceil$)

(117)
$$\frac{\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)} \sum_{\substack{k=-\infty \\ :|nx-k|\geq n^{1-\beta}}}^{\infty} \Phi(|nx-k|)\,|nx-k|^{N} \leq$$

$$\frac{\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda} \sum_{\substack{k=-\infty \\ :|nx-k|\geq n^{1-\beta}}}^{\infty} e^{-2\lambda|nx-k|}\,|nx-k|^{N} \leq \text{(as earlier)}$$

(118)
$$\leq \left(\frac{\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)}\right)\frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

Hence, it holds

(119)
$$|R_{1n}(x)|\,|_{\left|x-\frac{k}{n}\right|\geq\frac{1}{n^{\beta}}} \leq$$

$$\left(\frac{\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)}\right)\frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

Furthermore, we get that

$$|R_{2n}(x)|\,|_{\left|x-\frac{k}{n}\right|\geq\frac{1}{n^{\beta}}} \leq$$

$$\frac{\|D_{x-}^{\alpha}f\|_{\infty,(-\infty,x])}}{\Gamma(\alpha+1)} \sum_{\substack{k=-\infty \\ :\left|x-\frac{k}{n}\right|\geq\frac{1}{n^{\beta}}}}^{\lfloor nx \rfloor} \Phi(nx-k)\left(x-\frac{k}{n}\right)^{\alpha} \leq$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x])}}{n^{\alpha}\Gamma(\alpha+1)} \quad \sum_{\substack{k=-\infty \\ :\,|nx-k|\geq n^{1-\beta}}}^{\infty} \Phi(|nx-k|)\,|nx-k|^{\alpha} \leq$$

(120)
$$\left(\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x])}}{n^{\alpha}\Gamma(\alpha+1)}\right) \quad \sum_{\substack{k=-\infty \\ :\,|nx-k|\geq n^{1-\beta}}}^{\infty} \Phi(|nx-k|)\,|nx-k|^{N} \leq \text{(as earlier)}$$

$$\leq \left(\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x])}}{n^{\alpha}\Gamma(\alpha+1)}\right) \frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}} 2e^{2\lambda}N!\,e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

That is

$$\left|R_{2n}(x)\right|\big|_{\left|x-\frac{k}{n}\right|\geq\frac{1}{n^{\beta}}} \leq$$

(121)
$$\left(\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x])}}{n^{\alpha}\Gamma(\alpha+1)}\right) \frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}} 2e^{2\lambda}N!\,e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

We have proved that

(122) $$\left|R_{n}(x)\right| \leq \frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)}\left[\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}\right]+$$

$$\frac{1}{n^{\alpha}\Gamma(\alpha+1)}\frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}}2e^{2\lambda}N!\,e^{-\lambda\left(n^{1-\beta}-1\right)}\left[\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}\right].$$

For large enough $n\in\mathbb{N}$, we derive that

(123) $$\left|R_{n}(x)\right| \leq \frac{2\left[\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}\right]}{\Gamma(\alpha+1)\,n^{\alpha\beta}}.$$

That is

(124) $$\left|R_{n}(x)\right| = O\left(\frac{1}{n^{\alpha\beta}}\right),$$

and

(125) $$\left|R_{n}(x)\right| = o(1).$$

And, letting $0<\varepsilon\leq\alpha$, we obtain

(126) $$\frac{\left|R_{n}(x)\right|}{\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right)} \leq \frac{2\left[\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}\right]}{\Gamma(\alpha+1)}\left(\frac{1}{n^{\beta\varepsilon}}\right)\to 0, \quad \text{as } n\to\infty.$$

I.e.

(127) $$\left|R_{n}(x)\right| = o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right),$$

proving the claim. $\qquad\square$

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