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PHASE-PORTRAIT OF A MODIFIED GENERALIZED LIÉNARD TYPE SYSTEM

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ABSTRACT. This work deals with a new class of generalized Liénard type systems of the form

$$\dot{x} = y - \lambda(A - R(y))F(x), \qquad \dot{y} = -g(x),$$

where A is a positive constant and $g(x)$ and $F(x)$ are the typical nonlinearities occurring in the study of the van der Pol equation. For the function $R(y)$ we assume $R(y) \approx |y|^p$. The study of the above model is motivated by recent works which already appeared in the literature for the case $A = 0$. In the present paper, we discuss the problem of existence, non-existence, uniqueness and multiplicity of the limit cycles, by moving the parameter A. We show that this case has a rich dynamics and, in particular, the presence of some new features which also put in evidence the existence of subtle relation between the distribution of the zeros of $F(x)$ and the existence/non-existence of limit cycles.

AMS (MOS) Subject Classification. 34C25, 34C05, 34C23.

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1. **Introduction**

The Liénard system

$$\begin{cases} \dot{x} = y - \lambda F(x) \\ \dot{y} = -g(x) \end{cases} \tag{1.1}$$

and its generalization can be considered as an inexhaustible source of interesting problems regarding the study of planar dynamical systems, such as: stability/instability of the origin [24, 17], existence and multiplicity of limit cycles [12, 28, 30], existence of unbounded separatrices and their properties [42, 46], just to mention some contributions in different directions, within a literature with thousands of entries. Here

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$\lambda > 0$ is a parameter which, in some cases, discriminates between existence and non-existence of limit cycles. In other situations, the existence/non-existence/multiplicity theorems are independent on the choice of λ and therefore, without loss of generality, we will assume $\lambda = 1$.

Results about the geometry associated with the trajectories of the autonomous system (1.1), besides their intrinsic interest, may also be exploited in order to develop the analysis of the periodic perturbations of (1.1), given, for instance, by

$$\begin{cases} \dot{x} = y - \lambda F(x) + E(t) \\ \dot{y} = -g(x) \end{cases} \tag{1.2}$$

and including topics like: the construction of positively invariant sets (trapping regions) [24, 53] where to apply the Brouwer fixed point theorem [5, 25, 45], the search of separatrices [36, 47, 48] which provide a-priori bounds that are crucial in the applications of topological degree techniques [2, 35], as well as dissipative-type dynamics [16, 26] or horseshoe-type configurations [6, 27, 40, 41] for the proof periodic points of the Poincaré map (i.e., subharmonic solutions) or more complex (chaotic) behavior [34, 37, 51]. With this respect, we would like also to recall the interesting historical survey [1] where it is clearly emphasized how the study of the autonomous and forced van der Pol–Liénard equations influenced the development of the theory of the dynamical systems of the XX Century.

Throughout the article we assume that $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous functions with

$$g(x)x > 0, \quad \text{for } \ x \neq 0.$$

This includes, as a very special case, $g(x) = Lx$ for $L > 0$. It will be also useful to introduce the function

$$G(x) := \int_0^x g(x)\, ds.$$

A typical condition assumed when dealing with (1.1) is given by $G(x) \to +\infty$ as $x \to \pm\infty$ (see [9, 13, 43] for more insights about this condition).

Concerning the map $F : \mathbb{R} \to \mathbb{R}$ (which is assumed to be locally Lipschitz, too), motivated by the van der Pol equation where

$$F(x) = \int_0^x f(s)\, ds = \frac{x^3}{3} - x,$$

with $f(x) = x^2 - 1$, the typical hypotheses are summarized by the following one that we denote as $(F0)$:

There exist $\alpha < 0 < \beta$ such that $F(\alpha) = F(0) = F(\beta) = 0$, with $F(x)x < 0$ for $x \in (\alpha, \beta) \setminus \{0\}$ and $F(x)x > 0$ for $x \notin [\alpha, \beta]$.

For F satisfying $(F0)$ and $g(x) = x$ the first general result about limit cycles for equation (1.1) dates back to the pioneering work of Liénard [29] [1] who proved that *if* $F(x)$ *is odd with* $F(x) \to +\infty$ *as* $x \to +\infty$ *and strictly monotone increasing for* $x > \beta$, *there is a unique asymptotically stable limit cycle which globally attracts all the points in* $\mathbb{R}^2 \setminus \{0\}$. On the other hand, Duff and Levinson in their classical article [10] found examples of multiplicity of limit cycles if the symmetry condition on $F(x)$ is removed (see [3, 4, 20, 50] and the references therein for a more comprehensive discussion of this topic from different points of view).

In view of the above mentioned works, as well as a large number of others in this area, a natural question arises about if and how existence/non-existence and multiplicity of limit cycles persists for more general planar systems of the form

$$\begin{cases} \dot{x} = y - \lambda\mathcal{F}(y, x) \\ \dot{y} = -g(x) \end{cases} \tag{1.3}$$

where $\mathcal{F}(y, x)$ keeps the sign properties described in $(F0)$ with respect to the x-variable. In the recent papers [51, 52] we initiated the study of a particular class of (1.3) where $\mathcal{F}$ splits as

$$\mathcal{F}(y, x) = B(y)F(x)$$

with $B(y)$ an even function. In particular, as a case study, we analyzed in detail the choice

$$B(y) = |y|^p, \quad p > 0$$

and, for the superlinear case ($p > 1$), the following result was obtained for system

$$\begin{cases} \dot{x} = y - \lambda|y|^pF(x) \\ \dot{y} = -g(x) \end{cases} \tag{1.4}$$

(see [52, Theorem 6]).

Theorem 1.1. *Suppose* $p > 1$. *Let* F *satisfy* $(F0)$ *and assume* $F(x)\mathrm{sign}(x) \to +\infty$ *as well as* $G(x) \to +\infty$, *for* $x \to \pm\infty$. *Then there exists* $\Lambda^* > 0$ *such that system* (1.4) *has at least a limit cycle for* $\lambda \in (0, \Lambda^*)$ *and no limit cycles for* $\lambda > \Lambda^*$.

The same result holds also for $B(y)$ behaving like $|y|^p$ ($p > 1$) for $x \to 0$ and $x \to \infty$. The typical choice of nonlinearities in [52] was given by $g(x) = x$ and $F(x) = x^3 - x$, leading to

$$\begin{cases} \dot{x} = y - \lambda|y|^p(x^3 - x) \\ \dot{y} = -x \end{cases} \tag{1.5}$$

In this case, some numerical estimates for Λ^* can be effectively produced. On the other hand, in [11] Gasull and Giacomini have provided some effective estimates of

[1] See also [33], for an interesting historical account on this subject.

the form $3\sqrt{2}(3/p)^{p/2}$ for an upper bound of the constant Λ^*, via a refined version of the Bendixson-Dulac method. This shows that for $p > 1$ large, the range of value of λ for which the existence of limit cycles is ensured, becomes extremely small. From this point of view, the case $p = 2$, looks particularly interesting, as it keeps the superlinear growth in the y-variable but allows to deal with $\lambda = 1$, too.

The present work is motivated by the intent of better understanding the nature of these classes of modified generalized Liénard systems, by exploring the effect on the existence and multiplicity of limit cycles for different choices of the function $B(y)$. We start as a case study with the system

$$\begin{cases} \dot{x} = y - (a^2 - y^2)(x^3 - x) \\ \dot{y} = -x \end{cases} \tag{1.6}$$

which, roughly speaking, when $a > 0$ is large and $|y|$ is small, should behave like the classical Liénard system, while, for $|a|$ small and $|y|$ large, should fit into (1.5) with time reversal [2]. With this respect, our main results in Section 2 and Section 3 provide the following corollary when applied to (1.6).

Theorem 1.2. *There is $a^* > 0$ such that for each $a \in (0, a^*)$, system* (1.6) *has at least two limit cycles: a first large unstable/repulsive limit cycle and a second small stable/attractive limit cycle. Moreover, there is $a^{**} \geq a^*$ such that for every $a > a^{**}$ the system has a unique stable/attractive limit cycle.*

Actually, under some additional assumptions on $F(x), R(y)$ and $g(x)$, our result can be generalized to the class of equations

$$\begin{cases} \dot{x} = y - \lambda(A - R(y))F(x) \\ \dot{y} = -g(x) \end{cases} \tag{1.7}$$

including (1.6) as a special case.

The plan of the paper is the following: In Section 2, starting from the value $A = 0$, we prove the existence of at least one unstable limit cycle, together with a second small stable limit cycle as soon as A starts to increase. Then, when A is large enough, the external unstable limit cycle disappears and we have at least one stable limit cycle. The presented figures in the first part of Section 2 treat the case study $R(y) = y^2$, $A = a^2$ $F(x) = x^3 - x$ and $g(x) = x$. One of the main results in this section provides the existence of a limit cycle for every $A > 0$ in the "symmetric case" $G(\alpha) = G(\beta)$ (cf. Theorem 2.7). This is achieved under very simple and basic assumptions on F, G and R. On the other hand, the situation is much more delicate

[2]For sake of simplicity we use $F(x) = x^3 - x$ instead of $F(x) = \frac{x^3}{3} - x$. In any case, one can easily pass from one choice of $F(x)$ to the other via a well known change of variable.

when $G(\alpha) \neq G(\beta)$. In this case, thanks to Theorem 2.13, we can ensure the existence of a limit cycle when $A > 0$ is sufficiently large. For the proof we show that when $A > 0$ grows enough, then there are separatrices from infinity in the x-direction (cf. Theorem 2.12). We think that this auxiliary result about separatrices may have an independent interest, in view of the recent research on this topic (see [42]). In Remark 2.9 and the final part of Section 2, where we treat in detail the case $p = 2$, we also briefly discuss our model in comparison to some similar features for the van der Pol equation (as, for instance, fast/slow motion) when the parameter A is large. This phenomenon is put in evidence in Figure 14 and Figure 15 at the end of Section 2.

Section 3 is devoted to the problem of uniqueness of the stable limit cycle. Here the situation is more delicate. Indeed, the result is correct for the case study, namely (1.6), while for the general case, some additional assumptions which links the values (A, p, λ) in (1.7) are required in order to achieve the result. In Section 4 we study the sublinear case, that is for $R(y) \approx |y|^p$ with $0 < p < 1$. Finally, in the Appendix, we provide an alternative proof of the existence of small limit cycle in the light of the Hopf bifurcation as well as a proof of the general uniqueness theorem due to Huang and Sun [21] and Kooij and Sun [22], adapted to our situation in Section 3.

The main results of this article are illustrated by several examples with numerical simulations. However, we stress the fast that all the existence/non-existence results, as well as the movement of the limit cycles are actually analytically proved.

2. **Main results**

In this section we study the planar system

$$\begin{cases} \dot{x} = y - \lambda(A - R(y))F(x) \\ \dot{y} = -g(x) \end{cases} \tag{2.1}$$

for $\lambda > 0$, where $g, F, R : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz-continuous functions with

$$g(x)x > 0 \text{ for } \ x \neq 0, \tag{$g0$}$$

together with

$$G(\pm\infty) = +\infty \quad \text{for } \ G(x) := \int_0^x g(s)\, ds \tag{$G0$}$$

and

$$R(y) > R(0) = 0 \text{ for all } \ y \neq 0. \tag{$R0$}$$

We will also suppose that $F(x)$ has the standard shape of the cubic function in the classical Liénard system, as recalled in the Introduction. More precisely, we will assume hypothesis $(F0)$ together with

$$\lim_{x\to-\infty} F(x) = -\infty, \qquad \lim_{x\to+\infty} F(x) = +\infty. \tag{$F1$}$$

Equation (2.1) defines a (local) dynamical system in the plane. Accordingly, we use the standard notation in this setting. In particular, given any initial point P_0, we denote by $P_0 t$ the point of the orbit of P_0 at the time t and by $\gamma^+(P_0)$, $\gamma^-(P_0)$ and $\gamma(P_0)$ the positive/negative semi-orbits and the orbit of P_0.

Remark 2.1. From $g(x)x > 0$ for $x \neq 0$ and $F(x)x < 0$ for $x \in (\alpha, \beta) \setminus \{0\}$, it follows that the origin is the only equilibrium point for system (2.1) and therefore, all the possible limit cycles, must surround the origin. Moreover, from condition $(R0)$ it also follows that, the origin is a repeller whenever $A > 0$. Indeed, if $A > 0$, by the continuity of $R(y)$ there is a $\delta = \delta_A > 0$ such that $A - R(y) > 0$ for all $y \in (-\delta, \delta)$. Hence, for the energy function

$$E(x,y) := \frac{1}{2}y^2 + G(x), \tag{2.2}$$

we have that

$$\dot{E} = -\lambda(A - R(y))F(x)g(x) > 0, \quad \forall\, (x,y) \in (\alpha,\beta) \times (-\delta,\delta),\ x \neq 0.$$

Thus E is a Lyapunov function for the reverse time in a neighborhood of the origin and the origin is a repeller. Actually, we can prove more. Indeed, if $P_0 = (x_0, y_0)$ is any initial point such that the level line $E(x,y) = E(x_0,y_0)$ is contained in $(\alpha,\beta)\times(-\delta,\delta)$, then $P_0 t \to (0,0)$ as $t \to -\infty$.

2.1. **Existence of an unstable limit cycle.** For our first result we consider the following additional conditions on $R(y)$, namely that $\dfrac{R(y)}{y}$ is strictly monotone increasing with

$$\lim_{y\to 0} \frac{R(y)}{y} = 0, \qquad \lim_{y\to\pm\infty} \frac{R(y)}{y} = \pm\infty. \tag{$R_{\sup}$}$$

We refer to $(R_{\sup})$ as to a *superlinear condition.* Under these hypotheses, the map $\eta : \mathbb{R} \to \mathbb{R}$ defined by

$$\eta(y) := \frac{R(y)}{y}, \text{ for } y \neq 0, \ \ \eta(0) = 0,$$

is a strictly increasing and continuous function with $\eta(\pm\infty) = \pm\infty$. Then, following [52], we introduce a last assumption of Karamata type (cf. [31]) in the form of

$$\exists\, q > 0 :\ \forall\, \sigma > 0, \quad \lim_{s\to+\infty} \frac{\eta^{-1}(\sigma s)}{-\eta^{-1}(-s)} = \sigma^q \tag{RV}$$

Notice that all the above assumptions on the function R are satisfied for for the natural choice $R(y) = |y|^p$ with $p > 1$. Then the following result holds.

Theorem 2.2. *Assume $(g0)-(G0)$, $(F0)-(F1)$ and $(R0)-(R_{\sup})-(RV)$. Then there is Λ^* such that for each $\lambda \in (0, \Lambda^*)$ system (2.1), with $A = 0$, has at least an unstable limit cycle while, for $\lambda > \Lambda^*$, all the nontrivial trajectories of (2.1), with $A = 0$, are unbounded in the past. Moreover, for each $\lambda \in (0, \Lambda^*)$, there is $A^*(\lambda) > 0$ such that,* *for each $A \in (0, A^*(\lambda))$, system* (2.1) *has at least an unstable limit cycle $\Gamma_{\lambda,A}$, such that for all the initial points in the region outside $\Gamma_{\lambda,A}$, the forward trajectories are unbounded.*

Proof. Let us consider at first the case $A = 0$, leading to system

$$\dot{x} = y + \lambda R(y)F(x), \qquad \dot{y} = -g(x). \tag{2.3}$$

Then, we apply the change of variables

$$t \mapsto -t, \quad y \mapsto -y, \tag{2.4}$$

which reverses the direction of time and mirrors the orbits with respect to the x-axis, thus transforming system (2.3) to

$$\dot{x} = y - \lambda \tilde{R}(y)F(x), \qquad \dot{y} = -g(x), \tag{2.5}$$

where we have set

$$\tilde{R}(y) = R(-y).$$

Now, system (2.5) satisfies the assumptions of [52, Theorem 6]. In that result it is proved that there is a $\Lambda^* > 0$ such that, for any $\lambda \in (0, \Lambda^*)$, system (2.5) has at least a limit cycle $\Gamma_{\lambda,0}$ which is asymptotically stable/attracting, while for $\lambda > \Lambda^*$, all the orbits (except the trivial one) are unbounded in the future. For $\lambda \in (0, \Lambda^*)$, the limit cycle may be obtained as the boundary of the region of repulsiveness of the origin, as it attracts all the points in a neighborhood of the origin. Indeed, by the sign of $R(y)$ and of $F(x)g(x)$ in (α, β) one can easily see that the origin is a repeller for (2.5).

Coming back from system (2.5) to system (2.3), in virtue of the change of variables (2.4), we have guaranteed the existence of a $\Lambda^* > 0$ such that there is at least one unstable/repelling limit cycle in the boundary of the region of attractivity of the origin when $\lambda \in (0, \Lambda^*)$, while for all $\lambda > \Lambda^*$ all the orbits, except the equilibrium point, are unbounded in the past. Figure 1 shows a portrait corresponding to the situation that we have proved. The simulation is made for (2.3) with $g(x) = x$, $F(x) = x^3 - x$ and $R(y) = y^2$, namely the case study. There is an external trajectory departing outside the limit cycle and moving far to it, to eventually become unbounded in the future and another trajectory departing inside the limit cycle that moves (very slowly) toward the origin. The trajectories wind in the clockwise sense around the origin.

Let now $\lambda \in (0, \Lambda^*)$ be fixed and let $\Gamma_{\lambda,0}$ be the repelling limit cycle of system (2.3), whose existence has been proved above. Now, dealing with (2.3), let $P_0 = (0, y_0)$ be a point inside the region bounded by the limit cycle $\Gamma_{\lambda,0}$ and let $P_1 = (0, y_1)$ be the first intersection (in backward time) of $\gamma^-(P_0)$ with the y-axis. Similarly, let $Q_0 = (0, w_0)$ be a point outside the region bounded by the limit cycle $\Gamma_{\lambda,0}$ (but sufficiently near the limit cycle) and let $Q_1 = (0, w_1)$ be the first intersection (in backward time) of $\gamma^-(Q_0)$ with the y-axis. We have $0 < y_0 < y_1 < w_1 < w_0$.

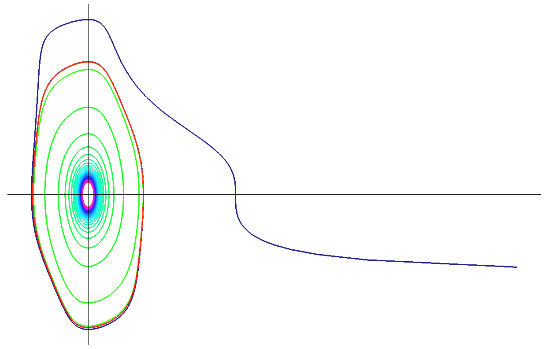

FIGURE 1. Example of a repelling limit cycle for system (2.3) with $\lambda = 1$.

If we move now the parameter A from $A = 0$ to a value $A = \varepsilon > 0$ for $\varepsilon > 0$ sufficiently small, by a continuity argument, we find that the negative semi-orbit of system (2.1) starting at P_0 will intersect again a first time the y-axis at a point $P_1^A = (0, y_1^A)$ above P_0 (and below Q_0), while the negative semi-orbit of the same system starting at Q_0 will intersect again a first time the y-axis at a point $Q_1^A = (0, w_1^A)$ below Q_0 and above P_1^A. In this manner, we obtain $0 < y_0 < y_1^A < w_1^A < w_0$ and the Poincaré-Bendixson theory guarantees the existence of an unstable limit cycle $\Gamma_{\lambda,A}$ crossing the y-axis at an intermediate point between P_0 and Q_0. This completes the proof of the theorem. □

2.2. **Existence of a second limit cycles for A small.**

Theorem 2.3. *Under the assumptions of Theorem 2.2, regarding g, F and R, assume that for some pair (λ, A) with $\lambda, A > 0$ there is an unstable limit cycle $\Gamma_{\lambda,A}$ to system* (2.1)*. Then, there exists a second (stable) limit cycle in the open region bounded by $\Gamma_{\lambda,A}$.*

Proof. We just use the fact that the origin is a repeller when $A > 0$, according to Remark 2.1, then the conclusion follows from the Poincaré-Bendixson theory because the orbits departing near the repelling limit cycle $\Gamma_{\lambda,A}$ wind in the clockwise sense toward the origin, while the orbits departing near the origin unwind with the energy function increasing its values as t increases. □

The next Figure 2 illustrates the presence of two limit cycles, according to Theorem 2.3. For the simulation, we have considered system (2.1) with $g(x) = x$, $F(x) = x^3 - x$ and $R(y) = y^2$, leading for $A = a^2$ to system

$$\dot{x} = y - \lambda(a^2 - y^2)(x^3 - x), \qquad \dot{y} = -x. \tag{2.6}$$

A simple numerical checking, as in [52, 51], shows that in this case the critical value Λ^* (computed for $a = 0$) satisfies $\Lambda^* > 1$. This is also consistent with the algebraic constant obtained by Gasull and Giacomini in [11]. Hence, for $\lambda = 1$ we have at least an unstable limit cycle, according to Theorem 2.2, provided that $a \in [0, a^*(\lambda))$. The second limit cycle appears as soon as $a > 0$. As the trajectories wind in the clockwise sense around the origin, we have at least an internal stable limit cycle and an external unstable one.

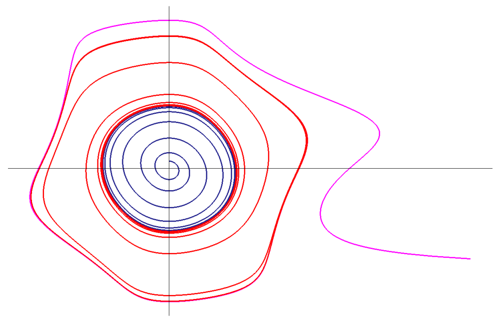

FIGURE 2. Example of two limit cycles for system (2.6) with $\lambda = 1$, $a = 1/2$. The trajectories wind in the clockwise sense around the origin.

The next Figure 3 shows how the external unstable limit cycle disappears and collapses to a separatrix, as an effect of being pushed out when the parameter $A = a^2$ increases.

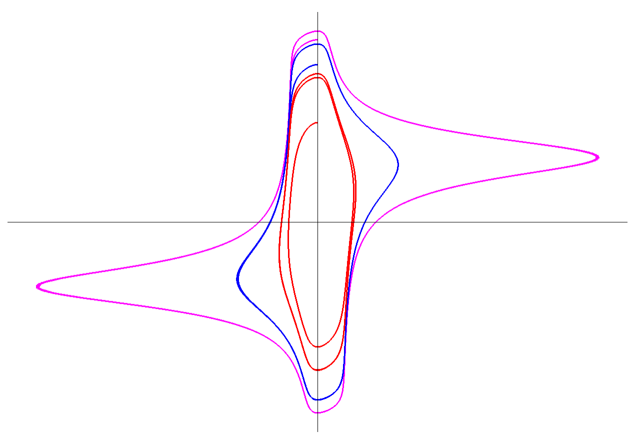

FIGURE 3. Example of moving the external limit cycle for system (2.6) with $\lambda = 1$, $a = 0.5$, $a = 0.7$ and $a = 0.78$.

In the above figures, to approach the external limit cycle, we start from an initial point inside the region bounded by the limit cycle (and not too far from it) and follow the trajectory in backward time.

As the parameter a increases, the external unstable limit cycle moves outside the limit cycles with smaller values of a. As a tends to some critical value $\hat{a}(\lambda)$, the external limit cycles are stretched in the x-direction until they disappear splitting into the union of some separatrices connecting points at infinity. This phenomenon will be proved in Section 3 where we deal also with the existence of separatrices, which, in turn, play a crucial role in our proof of the existence of limit cycles (see Theorem 2.13 and Theorem 2.15). Our result about unbounded trajectories in Theorem 2.12 may have some independent interest in view of recent results involving the studies of separatrices for Liénard type systems (see, for instance [42] and the references therein).

2.3. **Existence of a stable limit cycle for A large. Preliminary analysis.** Our aim now is to study the system (1.7) (or equivalently (2.1)) when the parameter A is not necessarily small. Throughout the foregoing analysis we will assume that g satisfies conditions $(g0)$ and $(G0)$, F satisfies $(F0)-(F1)$ and, moreover, as in the classical Liénard case,

- *$F(x)$ is increasing on $(-\infty,\alpha)$ and on $(\beta,+\infty)$* $(F2)$

and we suppose also that

- *R is strictly monotone decreasing on $(-\infty,0)$ and strictly monotone increasing on $(0,+\infty)$, with $R(y)\to+\infty$ as $y\to\pm\infty$.* $(R1)$

With these assumptions $R(y)$ has a strict absolute minimum point at $y=0$. Without loss of generality, we can suppose $R(0)=0$ (in order to have condition $R(0)$ satisfied). Indeed, if necessary, we can add and subtract $R(0)$ and deal with the new parameter $A_1=A+R(0)$ in the equation (this does not effect our results, as we are interested in the study of the equation for A large).

As we have already observed, the origin is the only equilibrium point of (2.1) and it is a source.

In this paragraph, all the numerical examples (which have only an illustrative nature) are made for the case $\lambda=1$. As we will see from our assumptions, this is not restrictive for the argument we are going to develop, because the presence of a possible parameter λ can be "incorporated" in the function F, by considering λF instead of F. This leads to the study of the equivalent system

$$\dot{x}=y-(A-R(y))\tilde{F}(x),\qquad \dot{y}=-g(x).$$

with $\tilde{F}=\lambda F$.

To start our analysis, we study at first the vertical isocline $\dot{x}=0$ of the system. This is described by the relation

$$\Psi_A(y)=\lambda F(x),\quad\text{for}\quad \Psi_A(y)=\frac{y}{A-R(y)}. \tag{2.7}$$

A typical graph of the function Ψ_A for $R(y)=|y|^p$ with $p>1$ (i.e., the superlinear case) is illustrated in Figure 4.

For each $A>0$, there are two solutions to the equation $R(y)=A$, that we denote by

$$\rho^-(A)<0<\rho^+(A),$$

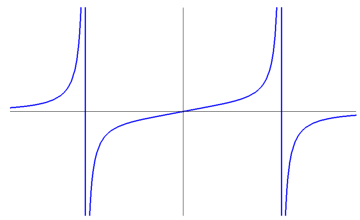

FIGURE 4. Typical behavior of the function Ψ_A in the superlinear case.

being

$$\rho^-(y) = \left(R|_{(-\infty,0]}\right)^{-1} \quad \text{and} \quad \rho^+(y) = \left(R|_{[0,+\infty)}\right)^{-1}$$

the inverse functions of R restricted to $(-\infty, 0]$ and $[0, +\infty)$, respectively.

Remark 2.4. A first observation that comes from the above positions is that *there is no cycle (that is a nontrivial periodic orbit) contained in the closed rectangle*

$$\mathcal{R} := [\alpha, \beta] \times [\rho^-(A), \rho^+(A)].$$

In fact, if we consider the energy $E(x, y)$ defined in (2.2) we find that $\dot{E}(x, y) > 0$ (except for the equilibrium and a discrete set of points), if evaluated along any orbit contained in $\mathcal{R}$. From $\oint_\gamma \dot{E} = 0$ along any cycle γ we obtain immediately our claim. In the special case in which condition

$$G(\alpha) = G(\beta), \quad \text{and} \quad R(-y) = R(y) \tag{2.8}$$

(see [4]) holds, we find that *any limit cycle must intersect both the vertical lines $x = \alpha$ and $x = \beta$, or at least one of the horizontal lines $y = \rho^-(A)$, $y = \rho^+(A)$,* (here we have $\rho^-(A) = -\rho^+(A)$). An illustration of this property is given in Figure 5. The limit cycle (in darker color) crosses the lines $x = -1$ and $y = -1$, but it does not crosses the lines $x = 4$ and $y = 1$. Notice that in this case, condition (2.8) is not satisfied. The simulation shows a rich structure of the system, with the presence of separatrices and escaping regions. This will be investigated with more details in Section 2.5.

The same considerations on the sign of $\dot{E}(x, y)$ from Remark 2.1 show that the domain of repulsiveness of the origin (which the basin of attraction of $(0, 0)$ for the inverted time) contains the open ellipsoidal region

$$\mathcal{D}_0 := \{(x, y) : \ E(x, y) < c_0\},$$

with

$$c_0 := \min\{G(\alpha), G(\beta), \frac{1}{2}\rho^+(A)^2, \frac{1}{2}\rho^-(A)^2\}. \tag{2.9}$$

As a next step, we invert the function Ψ_A on the intervals $(-\infty, \rho^-(A))$, $(\rho^-(A), \rho^+(A))$ and $(\rho^+(A), +\infty)$, respectively. Notice that, by very natural assumptions on $R(y)$, the function Ψ_A can be always inverted on $(\rho^-(A), \rho^+(A))$, while this is not always true for the other intervals. However, this is still true for the superlinear case $R(y) = |y|^p$

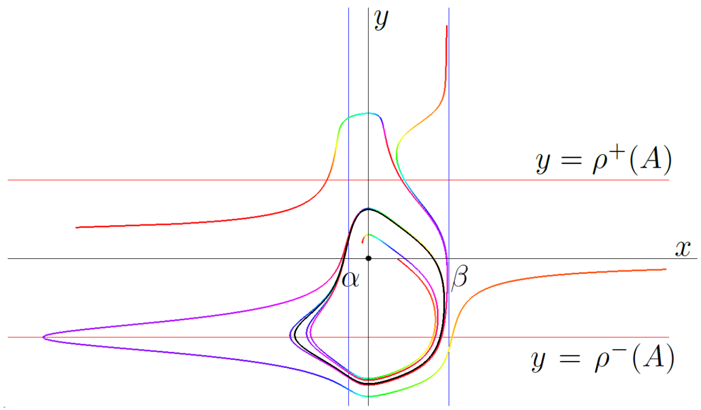

FIGURE 5. The figure shows the occurrence of a limit cycle for system (2.1), with $\lambda = 1$, $A = 4$, $R(y) = y^2$, $F(x) = x(x+1)(x-4)/10$ and $g(x) = x$.

for $p > 1$, as one can easily check by computing the derivative of Ψ_A. More precisely, if we assume that $R(y)$ is differentiable for $y \neq 0$, we find that the derivative of Ψ_A has the same sign as the function

$$N_A(y) := A - R(y) + yR'(y), \quad (A > 0, y \neq 0),$$

Thus, for $R(y) = |y|^p$, with $p \geq 1$, the function Ψ_A is always invertible in each interval of its domain. Moreover, for $p > 1$, Ψ_A is asymptotic to zero as $y \to \pm\infty$ and the graph is like that of Figure 4. On the other hand, for $0 < p < 1$, the function Ψ_A is invertible on $(\rho^-(A), \rho^+(A))$, provided that A is sufficiently large, but it is not invertible on $(-\infty, \rho^-(A))$ and $(\rho^+(A), +\infty)$. To invert the function in these intervals we need to further spilt these intervals into suitable subintervals where Ψ_A is strictly monotone.

Assuming Ψ_A invertible in all the three open intervals in which its domain is split, we can represent the vertical isocline in (2.7) as the union of the graphs of five curves. More precisely, the inversion of Ψ_A on the open interval $(\rho^-(A), \rho^+(A))$ is given by

$$y = \phi_A(x) := \left(\Psi_A|_{(\rho^-(A),\rho^+(A))}\right)^{-1}(\lambda F(x)), \tag{2.10}$$

which has a shape similar to $\arctan(\lambda F(x))$. The relations

$$\Psi_A(y) = \lambda F(x), \quad \text{for} \ \ y < \rho^-(A), \tag{2.11}$$

$$\Psi_A(y) = \lambda F(x), \quad \text{for} \ \ y > \rho^+(A), \tag{2.12}$$

typically provide other four curves. In fact, with reference to Figure 4, $\Psi_A(y) = \lambda F(x)$ for $y \in (\rho^+(A), +\infty)$ involves the intervals $(-\infty, \alpha)$ and $(0, \beta)$ for the x-variable where $F(x) < 0$, while $\Psi_A(y) = \lambda F(x)$ for $y \in (-\infty, \rho^-(A))$ involves the intervals $(\beta, +\infty)$ and $(\alpha, 0)$ for the x-variable where $F(x) > 0$. These four curves correspond to the boundaries of the regions $\mathcal{W}^+$ and $\mathcal{V}^+$ for (2.11) and the boundaries of $\mathcal{W}^-$ and $\mathcal{V}^-$ for (2.12), as depicted in Figure 7.

As observed above, we cannot always invert the last two relations (2.11) and (2.11) unless some extra hypotheses on $R(y)$ are assumed (as recalled in the superlinear case). In particular, for $R(y) = |y|^p$ with $0 < p < 1$, we have that $\Psi_A(y) \to \mp\infty$ as $y \to \pm\infty$. Since this section is mainly focused on the study of the superlinear case $R(y) = |y|^p$ with $p > 1$, the shape of the five branches of the isocline is the same as illustrated in Figure 6 and Figure 7.

As we will show in the foregoing analysis, the behavior in the horizontal strip

$$\mathcal{S} := \mathbb{R} \times (\rho^-(A), \rho^+(A))$$

is crucial.

For the map ϕ_A defined in (2.10), the following properties are satisfied.

$$\phi_A : \mathbb{R} \to (\rho^-(A), \rho^+(A)),$$

$$\lim_{x\to-\infty} \phi_A(x) = \rho^-(A), \qquad \lim_{x\to+\infty} \phi_A(x) = \rho^+(A).$$

Moreover, $\phi_A(x)$ has the same sign and monotonicity properties of $F(x)$. In particular, $\phi_A(x)$ is strictly increasing on $(-\infty, \alpha)$ and on $(\beta, +\infty)$, with $\phi_A(\alpha) = \phi_A(\beta) = 0$. On the graph of the function ϕ_A we have $\dot{x} = 0$ and, moreover, $\dot{y} < 0$ for $x > 0$ and $\dot{y} > 0$ for $x < 0$.

Finally, we also observe that

$$\lim_{A\to+\infty} \rho^+(A) = +\infty, \qquad \lim_{A\to+\infty} \rho^-(A) = -\infty.$$

From the study of the behavior of the solutions of the equation

$$\Psi_A(y) = k \in \mathbb{R} \setminus \{0\} \tag{2.13}$$

with respect to the parameter A, one easily detects the movement of the isocline as A varies. More precisely, as A grows, the components of the isocline (except for the points where $F(x) = 0$) move closer to the horizontal asymptotes $y = \rho^\pm(A)$.

To put our assertion into a formal statement (and considering still the superlinear case), we observe that for $k \neq 0$, the equation (2.13) has precisely two solutions $y'(k) < \rho^-(A)$, $y''(k) < \rho^+(A)$ if $k > 0$ and two solutions $y'(k) > \rho^-(A)$, $y''(k) > \rho^+(A)$ if $k < 0$ (this is also evident if one looks for the intersections of the graph $z = \Psi_A(y)$ with the horizontal line $z = k$ in Fig. 4). Now we can express $y'(k)$ and $y”(k)$ as $\rho^\pm(A)\sigma$ or $\rho^\pm(A) \pm \delta$, with an obvious choice if the signs and with σ as well as δ depending on k. For instance, if $k > 0$, then $y”(k) = \rho^+(A) - \delta$, while, if $k < 0$, then $y”(k) = \rho^+(A) + \delta$. Then the following result holds.

Proposition 2.5. Assume $R(y) = |y|^p$ with $p > 1$. Then $\sigma \to 1$ as $A \to +\infty$. Moreover, if $p > 2$, then $\delta \to 0$ as $A \to +\infty$.

Proof. Just to fix one of the four possible possibilities, we discuss the case $k < 0$ and $y(k) := y"(k) = \rho^+(A)\sigma$ with $\sigma > 1$. From (2.13) and setting $s := \rho^+(A)$, with $A = R(s) = s^p$, we have $s\sigma = (R(s\sigma) - s^p)|k| = s^p(\sigma^p - 1)|k|$, so that $s^{\frac{p-1}{p}}\sigma = (\sigma^p - 1)|k|$ and therefore, $\sigma \to 1$ as $s \to +\infty$ (for $p > 1$).

On the other hand, if we write $y(k) := y"(k) = \rho^+(A) + \delta$, with $\delta > 0$, from (2.13) and setting $s := \rho^+(A)$ as above, we have $s + \delta = (R(s + \delta) - s^p)|k| = s^p(1+(\delta/s))^{1/p}-1)|k|$. Then, by $(1+\varepsilon)^{1/p}-1 \approx p\varepsilon$, we find that $1+(\delta/s) \approx ps^{p-2}\delta|k|$, as $s \to +\infty$. From this and using the first part of the result which ensured that $\delta/s \to 0$, we conclude that $\delta \to 0$ as $s \to +\infty$ if $p > 2$. □

The same result holds if $R(y) \approx |y|^p$.

The effect of moving the parameter A is shown in Figure 6. This will be crucial in the proof of Theorem 2.13.

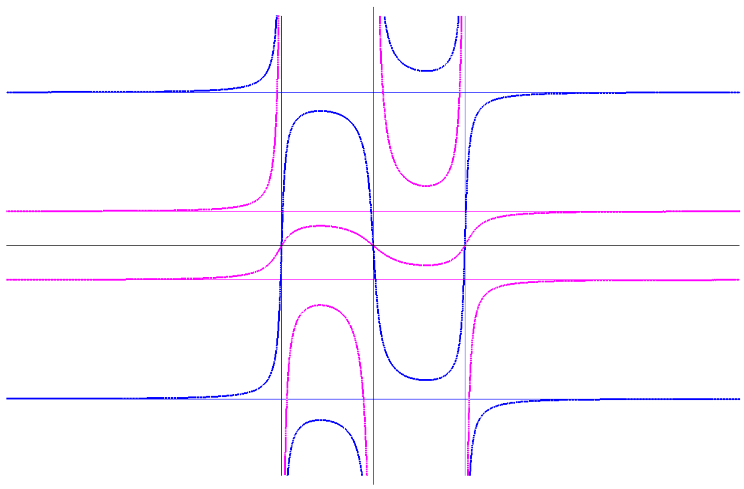

FIGURE 6. Moving the vertical isocline $\dot{x} = 0$, by changing the parameter A. For the figure have considered $R(y) = y^2$, $\lambda = 1$, $F(x) = x^3 - x$ and $g(x) = x$, for $A = 5$ and $A = 100$, respectively.

The vector field

$$\vec{Z} := (X, Y) = (y - (A - R(y))\lambda F(x), -g(x))$$

points outward at the boundary of the region

$$\mathcal{V}^- := \{(x, y) : \alpha < x < 0, y < \rho^-(A), \Psi_A(y) \leq F(x)\}$$

and inward at the boundary of the region

$$\mathcal{W}^- := \{(x, y) : x > \beta, y < \rho^-(A), \Psi_A(y) \leq \lambda F(x)\}.$$

Hence $\mathcal{V}^-$ is negatively invariant, while $\mathcal{W}^-$ is positively invariant. Symmetrically, $\vec{Z}$ points outward at the boundary of the region

$$\mathcal{V}^+ := \{(x, y) : 0 < x < \beta, y > \rho^+(A), \Psi_A(y) \geq \lambda F(x)\}$$

and inward at the boundary of the region

$$\mathcal{W}^+ := \{(x,y) : x < \alpha, y > \rho^+(A), \ \Psi_A(y) \geq \lambda F(x)\}.$$

Hence $\mathcal{V}^+$ is negatively invariant, while $\mathcal{W}^+$ is positively invariant.

A summary of all the above properties is illustrated by Figure 7. The arrows denote the direction of the vector field at the points of the vertical isocline.

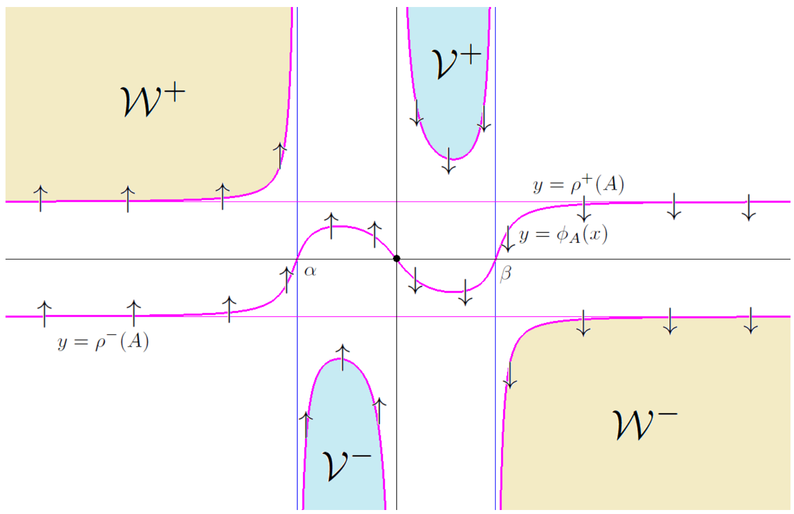

FIGURE 7. Typical portrait of the vertical isocline $\dot{x} = 0$ with the negatively (resp. positively) invariant regions $\mathcal{V}^\pm$ (resp. $\mathcal{W}^\pm$). The figure is made for $R(y) = y^2$.

We also denote by x_M a point in $(\alpha, 0)$ such that

$$F(x_M) = \max_{x \in [\alpha, 0]} F(x) = \max_{x \leq \beta} F(x) > 0$$

and by x_m a point in $(0, \beta)$ such that

$$F(x_m) = \min_{x \in [0, \beta]} F(x) = \min_{x \geq \alpha} F(x) < 0.$$

Remark 2.6. An elementary analysis of the vector field shows that for any initial point $P_0 = (x_0, y_0)$, with $x_0 \geq 0$ and $\sqrt{2c_0} \leq y_0 \leq \rho^+(A)$ the positive semi-orbit $\gamma^+(P_0)$ crosses the vertical isocline $y = \phi_A(x)$ at some point (x_1, y_1) with $x_1 > x_0$. The constant c_0 is the one defined in (2.9). A symmetrical result holds for $x_0 \leq 0$ and $\rho^-(A) \leq y_0 \leq -\sqrt{2c_0}$. In this case $\gamma^+(P_0)$ crosses the vertical isocline $y = \phi_A(x)$ at some point (x_1, y_1) with $x_1 < x_0$.

We stress the fact that the above observation holds for initial points inside the horizontal strip $\mathcal{S} = \mathbb{R} \times (\rho^+(A), \rho^+(A))$. Figure 8 shows a limiting example, given by positive semi-orbits which are deformed in the x-direction before intersecting the vertical isocline and tending asymptotically to a limit cycle. For the considered initial points, the orbits turn out to be unbounded in the past (in the x-component). If we take initial points a little farther from the origin, the orbit will escape (in the x-component) also in the future and do not cross the vertical isocline. Notice also that in our example the limit cycle is contained in the horizontal strip $\mathcal{S}$, while it intersects both the vertical lines $x = \alpha$ and $x = \beta$. This is consistent with Remark 2.4.

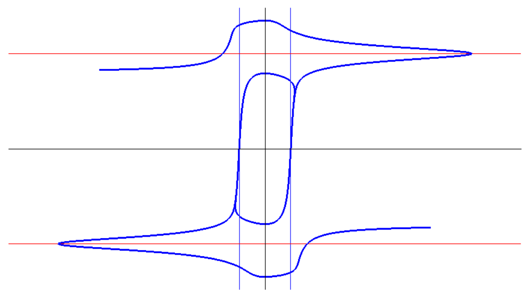

FIGURE 8. For the figure have considered $R(y) = y^2$, $F(x) = x^3 - x$ and $g(x) = x$, for $A = 6$ and $\lambda = 1$. The orbit paths are obtained for the initial points $(0, \pm 3.3)$ and the time-interval $[-0.96, 20]$.

2.4. **Existence of a stable limit cycle for every $A > 0$. The symmetric case.** In what follows, we show that an elementary argument allows to prove that for every $A > 0$ there is at least a stable limit cycle provided that, besides the natural assumptions $(g0)-(G0)$, $(F0)$ and $(R0)-(R1)$, also a symmetric condition, inspired by [4], is satisfied. More precisely, throughout this section, we suppose that

$$(S) \qquad G(\alpha) = G(\beta), \quad \text{and} \quad \rho^-(A) = -\rho^+(A)$$

holds. The first assumption in (S) corresponds to condition (D) in [4, Theorem 1.1], which plays a crucial role in the proof of the uniqueness of limit cycles. Observe that (S) is satisfied when (2.8) is assumed and, clearly, when g, F are odd functions and R is even. Then the following result holds.

Theorem 2.7. *Assume $(g0)-(G0)$, $(F0)$, $(R0)-(R1)$ and (S). Then, for every $\lambda > 0$, $A > 0$, system (2.1) has a limit cycle which is stable from the exterior.*

Proof. Let us consider the energy function for the associated Duffing equation

$$E(x, y) := \frac{1}{2}y^2 + G(x)$$

already introduced in (2.2). By the symmetry condition (S), the level line $E(x, y) = c_1$ for

$$c_1 := \frac{1}{2}\rho^+(A)^2 + G(\beta)$$

passes through the four points

$$(\alpha, \rho^-(A)), \quad (\beta, \rho^-(A)), \quad (\beta, \rho^+(A)), \quad (\alpha, \rho^+(A))$$

and contains the rectangle $\mathcal{R} = [\alpha, \beta] \times [\rho^-(A), \rho^+(A)]$. Moreover, $|y| > \rho^+(A) = -\rho^-(A)$ for $\alpha < x < \beta$ and also $\rho^-(A) < y < \rho^+(A)$ for $x < \alpha$ and for $x > \beta$. As a consequence,

$$\dot{E} = -\lambda(A - R(y))\lambda F(x)g(x) \leq 0, \quad \text{for} \;\; E(x.y) = c_1$$

with the strict inequality except for the four points listed above and the two points where the level line $E = c_1$ intersects the y-axis. On the other hand, as we have previously proved (see (2.9)) $\dot{E} \geq 0$ for

$$E(x, y) = c_0 := \min\{G(\alpha) = G(\beta), \frac{1}{2}\rho^+(A)^2\}$$

with the strict inequality except for $(\alpha, 0)$, $(\beta, 0)$ and the two points where the level line $E = c_0$ intersects the y-axis.

In this manner we conclude that the annular region

$$\mathcal{A} := \{(x, y) : c_0 \leq E(x, y) \leq c_1\}$$

is an equilibrium-free positively invariant set and therefore it must contain (cf. [15, p. 57]) at least a limit cycle which is stable from the exterior and a limit cycle which is stable from the interior (the two cycles possibly coinciding to a unique asymptotically stable limit cycle). Actually, in the polynomial case, $\mathcal{A}$ contains at least one asymptotically stable limit cycle (see [38, p. 252]). □

Remark 2.8. It may be interesting to observe that the positively invariant annular region $\mathcal{A}$ may overlap with the negatively invariant sets $\mathcal{V}^{\pm}$. Since the complement set of a negatively invariant set is positively invariant, we find that the limit cycles whose existence is proved in Theorem 2.7 are contained in the reduced annulus

$$\mathcal{A}' := \mathcal{A} \setminus (\mathcal{V}^+ \cup \mathcal{V}^-).$$

The next Figure 9 provides a specific example of this case with $\mathcal{A}$ and $\mathcal{V}^{\pm}$ overlapping and $\mathcal{A}'$ strictly contained in $\mathcal{A}$.

The simulation is made for (2.1) with $\lambda = 1$, $g(x) = x$, $F(x) = (x^2 - 4)x$, $R(y) = y^2$ and $A = 1$.

Remark 2.9. The are some other geometric features which are apparent from Figure 9. The first aspect to notice is that the limit cycle does not crosses the vertical lines $x = \alpha$ and $x = \beta$ but, consistently with Remark 2.1, it crosses the horizontal asymptotes $y = \rho^{\pm}(A)$. A second evident aspect is due to the fact that the limit cycle looks somehow deformed and it seems to move toward the boundaries of $\mathcal{V}^{\pm}$. These facts looks somehow in contrast with the shape of the limit cycle in Figure 8.

If we compare the functions and coefficients chosen in these two examples, we have that $g(x) = x$ and $R(y) = y^2$ in both the case, while $A = 6$, $F(x) = (x^2 - 1)x$ in

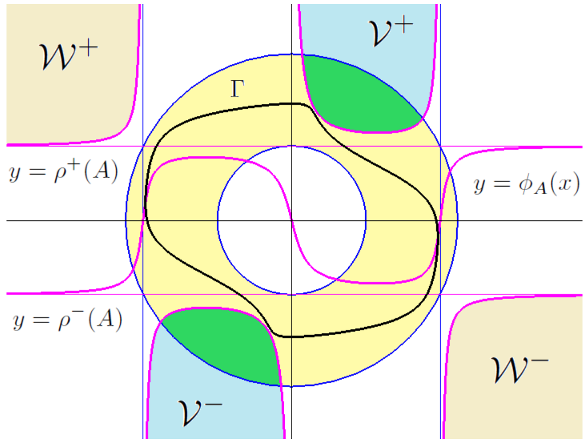

FIGURE 9. The figure shows an example in which the annular region $\mathcal{A}$ constructed in the proof of Theorem 2.7 overlaps with $\mathcal{V}^{\pm}$. The resulting limit cycle Γ is contained in $\mathcal{A}'$.

the simulation of Figure 8 and $A = 1$, $F(x) = (x^2 - 4)x$, in the simulation of 9. To understand the different outcome, we observe that, given a system of the form

$$\begin{cases} \dot{x} = y - \lambda(A - |y|^p)(x^2 - m^2)x \\ \dot{y} = -x \end{cases} \tag{2.14}$$

via the change of variable $x = mu$, $y = mv$, we transform it to the new system

$$\begin{cases} \dot{u} = v - \lambda_1(A_1 - |v|^p)(u^2 - 1)u \\ \dot{v} = -u \end{cases} \tag{2.15}$$

with

$$A_1 = \frac{A}{|m|^p} \quad \text{and} \quad \lambda_1 = |m|^{p+2}.$$

This shows that enlarging the distance between the two zeros of $F(x)$ produces the effect of reducing the constant A (and this explains why in Figure 9 the limit cycle crosses the horizontal asymptotes $y = \rho^{\pm}(A)$). On the other hand, the constant λ becomes larger and in this case we have an effect similar to the to the deformation of the limit cycle for fast/slow motions in the classical van der Pol–Liénard system when the parameter λ grows (cf. [15, Figure II.1.6]).

We end this paragraph with a corollary summarizing the content of Theorem 1.1 (taken from [52]), Theorem 2.2 and Theorem 2.7 applied to the differential system

$$\dot{x} = y - \lambda(A - |y|^p)(x^3 - x), \qquad \dot{x} = y, \tag{2.16}$$

where $F(x) = x(x^2 - 1)$ has the classical shape of the van der Pol equation in the Liénard plane.

Corollary 2.10. *Assume $p > 1$. Then, the following conclusion holds.*

- *For every* $\lambda > 0$ *and* $A > 0$, (2.16) *has at least a stable limit cycle* $\Gamma' = \Gamma'_{\lambda,A}$.
- *There exists* Λ^* *such that for each* $\lambda \in (0, \Lambda^*)$, *there is* $A^* = A^*(\lambda)$, *such that for each* $A \in (0, A^*)$ *there is at least a second unstable limit cycle* $\Gamma'' = \Gamma''_{\lambda,A}$, *with* Γ'' *surrounding* Γ'.

2.5. **Existence and non-existence of limit cycles for** A **large. The asymmetric case.** When the symmetry condition (S) is not satisfied, we cannot construct the outer boundary of the positively invariant annular region as in the proof of Theorem 2.7. Just to fix the ideas, we assume that $R(y)$ is even, so that $\rho^-(A) = -\rho^+(A)$ but

$$G(\alpha) > G(\beta).$$

Now, if we consider for the energy $E(x, y)$ defined in (2.2), the level line Γ_α passing through $(\alpha, \rho^\pm(A))$ and the level line Γ_β passing through $(\beta, \rho^\pm(A))$, we have that Γ_α encircles Γ_β. Moreover, $\dot{E}(x, y) > 0$ on the points of Γ_α for $x > \beta$ and $|y| > \rho^+(A) = |\rho^-(A)|$. Therefore, if we start a trajectory $\gamma^+(P)$ at the point $P = (\alpha, \rho^+(A))$, we have that this trajectory is bounded by Γ_α until it reaches the vertical line $x = \beta$. After this moment, we do not have a way to control $\gamma^+(P)$ in terms of Γ_α and it could happen that $\gamma^+(P)$ unwinds and become eventually unbounded.

Example 2.11. Let us consider the case

$$g(x) = x, \quad R(y) = y^2, \quad \text{and} \;\; A = 1, \quad \lambda = 1,$$

for the function

$$F(x) = x(x - \alpha)(x - \beta), \quad \text{with} \;\; \alpha = -4, \; \beta = \frac{1}{2}\,.$$

The asymmetry of the branch of the vertical isocline in the strip $\mathcal{S}$, given by $y = \phi_A(x)$, as well as of the sets $\mathcal{V}^\pm$ and $\mathcal{W}^\pm$ is quite evident from Figure 10. In Theorem 2.7 and the example illustrated in Figure 9, for the same choice of $g(x)$, $R(y)$ and A, we had provided the existence of a limit cycle in the case $\alpha = -2 = -\beta$. Now, the level lines passing through $(\beta, \rho^\pm(A))$ and $(\alpha, \rho^\pm(A))$ are not the same and do not bound positively invariant regions. In particular, the vector field $\vec{Z}$ is directed outwardly on Γ_α at the points (x, y) where $x > \beta$ and $|y| > \rho^+(A)$. Similarly, $\vec{Z}$ points outwardly on Γ_β at the points (x, y) where $\alpha < x < 0$ and $|y| < \rho^+(A)$. In Figure 10 we have also considered three circumferences around the origin that are (from the smaller to the larger): the level line $E(x, y) = c_0$, according to (2.9), as well as Γ_β and Γ_α. Furthermore, we have put in evidence a neighborhood of the origin U_0 bounded by $E(x, y) = c_0$ such that any point of $U_0 \setminus \{0\}$ exit U_0 after some time.

The next simulation in Figure 11 gives evidence of the non-existence of cycles, because it shows the presence of a trajectory γ which connects the origin to infinity. Since any cycle must encircle the origin, it should cross the orbit γ, which is impossible. In the simulation we take as initial point $P := (0, 0.1)$ which belongs to region

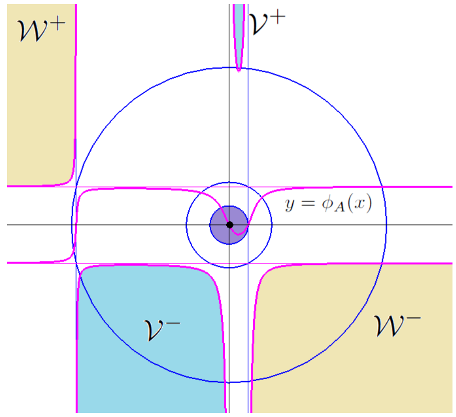

FIGURE 10. Portrait of the vertical isocline and the regions $\mathcal{V}^{\pm}$ and $\mathcal{W}^{\pm}$ in the asymmetric case $\dot{x} = y - (1-y^2)(x+4)(x-\frac{1}{2})x$, $\dot{y} = -x$.

of repulsiveness of the origin. The trajectory $\gamma = \gamma(P)$ is drawn for the time interval $[-3, 10.1545]$. It is clear that after a few turns around the origin the orbit becomes ultimately unbounded in the positive x-direction. Indeed, at $T = 10.156$ and beyond the software stops to produce a portrait, either because the object is too large or by overflow.

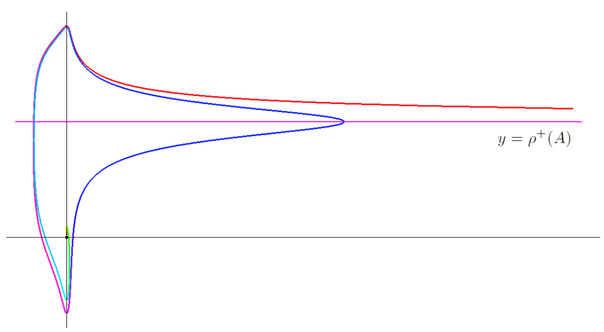

FIGURE 11. Example of an orbit approaching the origin in the past and becoming unbounded in forward time.

In view of these difficulties, we need to try a different strategy and, instead of looking for an energy level line bounding the trajectories, we search for the existence of separatrices.

Our first main result now reads as follows.

Theorem 2.12. *Assume*

$$\left| \int^{\pm\infty} \frac{g(x)}{F(x)}\, dx \right| < \infty. \tag{gF}$$

Then, for every $\lambda > 0$, there are a constant $A^{**} = A^{**}(\lambda) > 0$ and $\hat{y}_0 < -1$, such that for each $A > A^{**}$ the negative semi-orbit of the point (β, y_0), with $\rho^-(A) < y_0 \leq \hat{y}_0$, *is unbounded in the x-component (for $x > 0$) and does not cross the vertical isocline. Symmetrically, there is $\check{y}_0 > 1$ such that the negative semi-orbit of the point (α, y_0), with $\check{y}_0 \leq y_0 < \rho^+(A)$, is unbounded in the x-component (for $x < 0$) and does not cross the vertical isocline.*

Proof. We split the proof into two steps. In what follows, the parameter $\lambda > 0$ is fixed once for all.
Step 1. We fix an abscissa $d_0 > \beta$ and consider a point $Q_1 = (d_0, y_1)$ with $y_1 \leq 0$. Our aim is to prove that the negative semi-orbit $\gamma^-(Q_1)$ is unbounded in the x-component and remains below the isocline $y = \phi_A(x)$. To achieve this result, we prove the following
Claim 1. Let $L > 0$ be fixed. Then there are $y_1^* < F(d_0)$ and $A_1^{**} = A_1^{**}(d_0, L)$ such that for each $y_{\leq} y_1^*$ and $A > A_1^{**}$ the semi-orbit $\gamma^-(Q_1)$ for $x \geq d_0$ is contained in the strip $[d_0, +\infty) \times [y_1, L)$.
To prove our assertion, we construct a barrier $y = \psi(x)$ by a smooth strictly increasing function $\psi : [d_0, +\infty) \to [\nu, +\infty)$, with $\lim_{x \to +\infty} \psi(x) \leq L$ and $\nu = \psi(d_0)$. This is obtained by requiring that the region $\{(x, y) : x \geq d_0\,,\, y \leq \psi(x)\}$ is negatively invariant and this, in turns, follows by the geometric condition that the vector field $\vec{Z}$ points outward at upper boundary of the region $y - \psi(x) \geq 0$. The geometric condition is now expressed by

$$\psi'(x)((A - R(y))\lambda F(x) - y) - g(x) \geq 0,\ \forall\, x \geq d_0, \quad \text{with} \quad y = \psi(x),$$

that can be equivalently written as

$$\psi'(x)(A - R(\psi(x)))\frac{\lambda F(x)}{g(x)} - 1 \geq \frac{\psi'(x)}{g(x)}\psi(x),\ \forall\, x \geq d_0. \tag{2.17}$$

We choose now

$$\psi(x) := \nu + \int_{d_0}^{x} \frac{g(\xi)}{\lambda F(\xi)}\, d\xi,$$

for a fixed constat $\nu \leq 0$, such that

$$\nu + \int_{d_0}^{+\infty} \frac{g(x)}{\lambda F(x)}\, dx \leq L, \tag{2.18}$$

In this manner, we have

$$\psi(d_0) = \nu \quad \text{and} \quad \psi(x) < L, \quad \forall\, x \geq d_0\,.$$

Then (2.17), after a simple manipulation, reduces to

$$(A - 1 - R(\psi(x)))\lambda F(x) \geq \psi(x),\ \forall\, x \geq d_0. \tag{2.19}$$

Using the fact that $F(x)$ is strictly monotone increasing on $(\beta, +\infty)$, we find that a sufficient condition for the validity of (2.19) is that

$$(A - 1 - R(\max\{|\nu|, L\}))\lambda F(d_0) \geq L.$$

This latter inequality is clearly satisfied provided that

$$A > A^{**}(d_0, L, \lambda) := \frac{L}{\lambda F(d_0)} + R((\max\{|\nu|, L\})) + 1. \tag{2.20}$$

As a last step, we take $y_1^* := \nu$ and the proof of the claim is complete.

From the proof of the claim and (2.18), it is clear that a suitable choice of y_1^* is given by

$$y_1^* = \min\left\{0, L - \int_{d_0}^{+\infty} \frac{g(x)}{\lambda F(x)}\, dx\right\}.$$

Now, to conclude the proof of this first part, we fix a constant $L > 0$ and A sufficiently large such that $\rho^+(A) > L > \psi(+\infty)$ and also $\phi_A(x) > \psi(x)$ for $x \geq d_0$. In this manner, the negative semi-orbit departing from $Q_1 = (d_0, y_1)$ with $y_1 \leq y_1^*$ remains bounded above by $y = \psi(x)$ and does not crosses the isocline $y = \phi_A(x)$.

Step 2. Consider the point $Q_1 = (d_0, y_1)$ as in Step 1, with

$$y_1 \leq \min\{y_1^*, -1\}.$$

On the interval $[\beta, d_0]$, the slope of the positive trajectory departing from Q_1 is given by

$$\frac{dy}{dx} = \frac{g(x)}{(A - R(y))\lambda F(x) - y} \leq \frac{g(x)}{-y} \leq g(x),$$

as long as $A > R(y)$ (which is guaranteed if A is sufficiently large). This also follows from the fact that $F(x) > 0$ for $x > \beta$. Then, if we denote by $Q_0 = (\beta, y_0)$ the intersection of the trajectory with $x = \beta$, we have that

$$y_1 - y_0 \leq K := G(d_0) - G(\beta),$$

that is

$$|y_0| = -y_0 \leq K + |y_1|.$$

Step 3. To conclude our proof, we just note that, having fixed d_0, L, y_1^*, we have found a constant A^{**}, such that for each $A > A^{**}$, we can find a negative semi-orbit departing from a point $\hat{Q}_0 = (\beta, \hat{y}_0)$ with

$$\rho^-(A) < \hat{y}_0 < -1, \tag{2.21}$$

such that $\gamma^-(\hat{Q}_0)$ is unbounded in the x-component and bounded above by the constant L in such a manner that $\gamma^-(\hat{Q}_0)$ does not cross the vertical isocline. Note that for A large, whatever $\hat{y}_0 < -1$ may be, we can always have (2.21) satisfied, due to the fact that $\rho^-(A) \to -\infty$ as $A \to +\infty$.

Clearly, the same conclusion holds for any initial point $Q_0 = (\beta, y_0)$ with

$$\rho^-(A) < y_0 \leq \hat{y}_0 < -1.$$

Finally, by a symmetric reasoning, we can prove that an analogous result holds on the negative semi-orbits departing at a point (α, y_0) for

$$\rho^-(A) > y_0 \geq \check{y}_0 > 1,$$

for $\check{y}_0$ a suitable constant determined in a way similar to $\hat{y}_0$. □

As a consequence of Theorem 2.12, we have the following crucial result.

Theorem 2.13. *Assume* (gF) *and* $R(y) \approx |y|^p$ *with* $p > 2$. *Then, for every* $\lambda > 0$, *there is a constant* $A^{**} = A^{**}(\lambda) > 0$, *such that for each* $A > A^{**}$ *system* (2.1) *has an outermost limit cycle which is asymptotically stable.*

Proof. By Theorem 2.12 we choose two negative semi-orbits $\gamma^-(Q')$ and $\gamma^-(Q'')$ with $Q' = (\beta, y_0')$ and $Q'' = (\alpha, y_0'')$ which are unbounded in the x-component and do not cross the vertical isocline $y = \phi_A(x)$. Here

$$\rho^-(A) < y_0' < -1 < 1 < y_0'' < \rho^+(A).$$

By the previous theorem, we know that this configuration is always achieved for A sufficiently large.

Now we take the initial point $P_0 = (\beta, y_0)$ with $y_0' < y_0 < 0$ and consider its positive semi-orbit $\gamma^+(P_0)$. A crucial step of the proof consists in showing that this trajectory remains bounded from above by $\gamma^-(Q'')$ and it it intersects the isocline $y = \phi_A(x)$ at a point P_1. In order to achieve this result, the position in the phase-plane of $\mathcal{V}^-$, which depends on the value of A (as it has been observed before and shown in Figure 6) plays a crucial role. This because when A grows larger the boundary of $\mathcal{V}^-$ approaches the asymptote $y = \rho^-(A)$ (here we apply also Proposition 2.5). As $\mathcal{V}^-$, which clearly is a barrier for the trajectory $\gamma^+(P_0)$, moves closer $y = \rho^-(A)$, then necessarily $\gamma^+(P_0)$ must intersects the asymptote $y = \rho^-(A)$ and, subsequently, the vertical isocline $y = \rho^-(A)$ (as observed in Remark 2.6).

Proceeding further as the time increases, after the point P_1, the positive semi-orbit $\gamma^+(P_0)$ is bounded from above by $\gamma^-(Q'')$, finds the set $\mathcal{V}^+$ as a barrier (with its boundary close to the asymptote $y = \rho^+(A)$) and its is bounded from below by $\gamma^-(Q')$; then it will intersect again the vertical isocline. Thus $\gamma^+(P_0)$ winds around the origin in the clockwise sense and is bounded in the future. Since the origin ia a repeller, we conclude that $\gamma^+(P_0)$ tends to a limit cycle. □

Figure 12 shows a configuration where we reproduce by a simulation the proof of Theorem 2.13. One negatively unbounded orbit in the fourth quadrant and another one in the second, force any trajectory departing from a point in the fourth quadrant above $\gamma^-(Q')$ to wind around the origin.

Our next goal is to prove that for $A > 0$ sufficiently large, any limit cycle is contained in the strip $\mathcal{S}$. We have numerical evidence that this property is not true

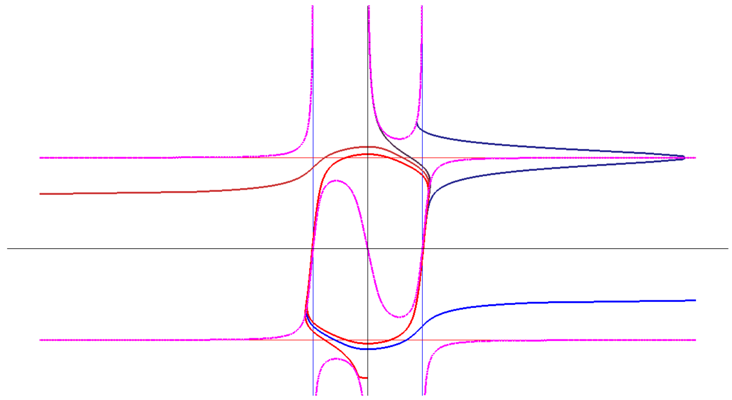

FIGURE 12. For the figure have considered $R(y) = y^4$, $F(x) = x^3 - x$ and $g(x) = x$, for $A = 4$ and $\lambda = 1$. Note that the resulting limit cycle is not contained in the horizontal strip $\mathcal{S} = \mathbb{R} \times (\rho^+(A), \rho^+(A))$.

for $R(y) = |y|^p$ when $p > 1$ is sufficiently large (see, for instance Figure 12). Therefore, we need to impose some extra conditions on $R(y)$. In the next theorems we make our computations for $R(y) = |y|^p$. We remark that the same results holds for $R(y) \approx |y|^p$ for $y \to \pm\infty$.

Theorem 2.14. *Assume (gF) and let $R(y) \approx |y|^p$ with $0 < p < 2$. Then, for every $\lambda > 0$, there is a constant $A^{**} = A^{**}(\lambda) > 0$, such that for each $A > A^{**}$, the negative semi-orbit of the point $(0, \rho^-(A))$, for $x > 0$, is contained in the strip $\mathcal{S}$, unbounded in the x-component and does not cross the vertical isocline. Symmetrically, the negative semi-orbit of the point $(0, \rho^+(A))$, for $x < 0$, is contained in the strip $\mathcal{S}$, unbounded in the x-component and does not cross the vertical isocline.*

Proof. Claim 1. Let $\lambda > 0$ be fixed. There is $A_1^{**} = A_1^{**}(\delta, d_0, \lambda)$ such that for each $A > A_1^{**}$ the arc of $\gamma^-(Q_0)$ for $0 < x \le d_0$ is contained in the rectangle $(0, d_0] \times (\rho^-(A), \rho^-(A - \delta))$.

To prove our assertion, we consider the slope of the trajectory

$$\frac{dy}{dx} = \frac{g(x)}{-y + (A - R(y))\lambda F(x)}$$

and observe that for $A > \delta$ and sufficiently large so that the denominator in the above fraction is positive, we have

$$0 < \frac{dy}{dx} \le \frac{g_{\max}(d_0)}{|\rho^-(A-\delta)| - (A - R(\rho^-(A-\delta)))|\lambda F(x_m)|} = \frac{g_{\max}(d_0)}{|\rho^-(A-\delta)| - \delta|\lambda F(x_m)|},$$

where we have set

$$g_{\max}(d_0) := \max_{0 \le x \le d_0} g(x).$$

Hence,

$$\rho^-(A) < y(s) = y(0) + \int_0^s \frac{dy}{dx}\, dx \le \rho^-(A) + \frac{s g_{\max}(d_0)}{|\rho^-(A-\delta)| - \delta|\lambda F(x_m)|}$$

for all $s > 0$ and such that the arc of trajectory $(s, y(s))$ remains in the rectangle $[0, d_0] \times [\rho^-(A), \rho^-(A-\delta)]$. Then, if

$$\frac{d_0 g_{\max}(d_0)}{|\rho^-(A-\delta)| - \delta|\lambda F(x_m)|} \leq |\rho^-(A)| - |\rho^-(A-\delta)|, \tag{2.22}$$

our claim is proved.

Now, if $R(y) \approx |y|^p$, we have $|\rho^-(A)| \approx A^{1/p}$ and $|\rho^-(A-\delta)| \approx (A-\delta)^{1/p}$ then (forgetting the term $\delta|\lambda F(x_m)|$ which is considered as negligible for $A \to +\infty$) we conclude that (2.22) is satisfied provided that

$$d_0 g_{\max}(d_0)(A-\delta)^{-1/p} < A^{1/p} - (A-\delta)^{1/p}. \tag{2.23}$$

This is true, for an appropriate choice of $\delta > 0$ and for $A > 0$ and large, if $p < 2$.

After the claim is verified, we just apply Theorem 2.12 and get the conclusion. The symmetric part is proved in the same manner. □

Combining Theorem 2.14 with Theorem 2.13 we obtain the following.

Theorem 2.15. *Assume* (gF) *and let* $R(y) \approx |y|^p$ *with* $0 < p < 2$. *Then, for every* $\lambda > 0$, *there is a constant* $A^{**} = A^{**}(\lambda) > 0$, *such that for each* $A > A^{**}$, *all the nontrivial periodic orbits of system* (2.1) *are contained in the strip* $\mathcal{S} = \mathbb{R} \times (\rho^-(A), \rho^+(A))$. *Moreover, for each* $A > A^{**}$ *there is at least a limit cycle.*

Proof. We just repeat the proof of Theorem 2.13, this time, considering the negative semi-orbits $\gamma^-(Q')$ and $\gamma^-(Q'')$ with $Q' = (0, \rho^-(A))$ and $Q'' = (0, \rho^+(A))$. By Theorem 2.15 we have that, for $A > 0$ sufficiently large, $\gamma^-(Q')$ intersects the vertical line $x = \beta$ at some point in the fourth quadrant and is unbounded in the x-component (for $x > \beta$), without intersecting the vertical isocline. Similarly, $\gamma^-(Q')$ intersects the vertical line $x = \alpha$ at some point in the second quadrant and is unbounded in the x-component (for $x < \alpha$), without intersecting the vertical isocline. If we take as initial point $P_0 = Q'$, we can repeat from now on the proof of Theorem 2.13 and show that the positive semi-orbit $\gamma^+(P_0)$ winds around the origin and tends to a limit cycle which, by construction, is bounded by $\gamma^-(Q')$ and $\gamma^-(Q'')$ and therefore is contained in the strip $\mathcal{S}$. □

Figure 13 show an example related to Theorem 2.15. The simulation deals with the "asymmetric" system

$$\dot{x} = y - (A - |y|^p)(x+4)(x - \tfrac{1}{2})x, \qquad \dot{y} = -x,$$

already considered in Figure 11 for the non-existence of a limit cycle in the case $A = 1$ and $p = 2$. In our next example we consider $A = 4$ and $p = \frac{2}{3}$ and a limit cycle appears in the strip $\mathcal{S}$. The limit cycle is also unique from the results in Section 3. Numerical simulations for the above choice of p, α, β show that there is $A^{**} \in (3.7, 3.8)$ such that for each $A > A^{**}$ the (unique) limit cycle is contained in the horizontal strip $\mathcal{S}$.

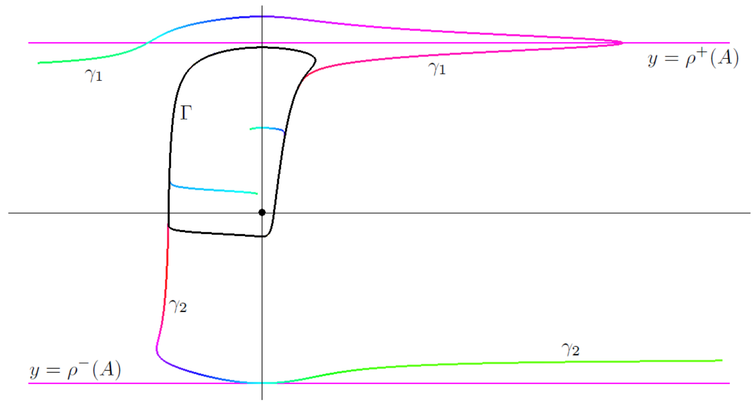

FIGURE 13. Existence of a limit cycle in the strip $\mathcal{S}$ in the asymmetric case for $1 < p < 2$ and $A > 0$ and large.

In Figure 13 it is also interesting to consider the obits γ_1 and γ_2 having the limit cycle Γ as ω-limit. The negative semi-orbits of these trajectories are unbounded in the x-directions and play the role bounding the positive semi-orbits departing from the negative x-axis and the positive x-axis, respectively, as prescribed by Theorem 2.12.

2.6. **The case $p = 2$..** The case $p = 2$ is more delicate and it depends on a balance between the growth of g, F and the distance between the zeros of F and the origin. Since Theorem 2.12 is still valid, here the key result is Theorem 2.14 and, if we repeat its proof for $p = 2$, we find that (2.22) and, especially (2.23) (which implies (2.22)), may hold or not depending on the choice of δ and $g(d_0)$. This latter constant, in turns, depends on the growth of g and the choice of d_0. However, in the applications of the theorem, d_0 is related to $\max\{|\alpha|, \beta\}$ and hence it depends on the distance of the origin from the zeros of $F(x)$. So, we can expect that if $\max\{|\alpha|, \beta\}$ is large, then the result fails.

The next examples give evidence of these assertions.

Figure 14 and Figure 15 provide examples related to Theorem 2.14 in the special case $p = 2$ and $\lambda = 1$. Both the simulations deal with the "symmetric" system

$$\dot{x} = y - (A - y^2)(x^2 - m^2)x, \qquad \dot{y} = -x, \tag{2.24}$$

already considered in Figure 9. In this case, the existence of a limit cycle is always guaranteed (for any value of $A > 0$) by Theorem 2.7. Accordingly, we focus our attention on the problem whether the limit cycle is contained in the strip $\mathcal{S}$. As we already remarked, the case $p = 2$ represents, in some sense, a crucial limiting case.

In Figure 14 we obtain a limit cycle Γ contained in the strip $\mathcal{S}$ (and this is the unique limit cycle, according to Theorem 3.1 in Section 3). The example considers the cases $A = 8$, $A = 16$ and $m = 1$, so that $\beta = -\alpha = 1$. Enlarging the value of A

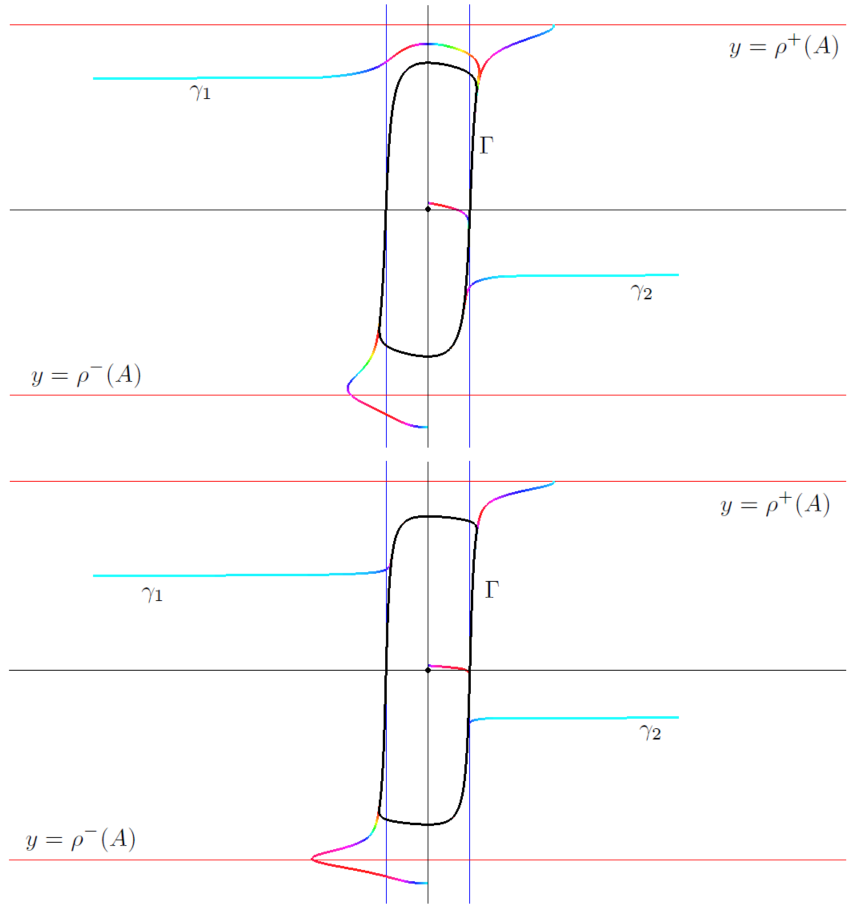

FIGURE 14. Existence of a limit cycle of system (2.24) in the strip $\mathcal{S}$ in the symmetric case $m = 1$ for $p = 2$. We have $A = 8$ in the upper panel and $A = 16$ in the lower panel. The phase-portraits are almost the same, from a qualitative point of view.

does not affect the qualitative properties of the limit cycle which remains inside the strip $\mathcal{S}$, except for the fact that the isocline $y = \phi_A(x)$ tends very quickly towards the asymptotes $y = \rho^{\pm}(x)$ and, moreover, its profile is almost vertical near the lines $x = \alpha$ and $x = \beta$. Consequently, as A increases, the limit cycle Γ follows slowly the isocline near $x = \alpha$ (respectively, near $x = \beta$) till to the point of maximum (respectively, minimum), then it jumps fast (and moving almost horizontally) to meet the isocline again near $x = \beta$ (respectively, near $x = \alpha$). In Figure 14 the orbits γ_1 and γ_2 have the limit cycle Γ as ω-limit set and are unbounded in the past, according to Theorem 2.12.

On the other hand, in Figure 15 we obtain a limit cycle Γ which is not contained in the strip $\mathcal{S}$. The example considers the cases $A = 1/2$, $A = 8$ and $m = 4$, so that $\beta = -\alpha = 4$. Enlarging the value of A does not affect the qualitative properties of the limit cycle which is not contained in the strip $\mathcal{S}$. As in the previous case, the isocline $y = \phi_A(x)$ tends very quickly towards the asymptotes $y = \rho^{\pm}(x)$ with its profile being vertical near the lines $x = \alpha$ and $x = \beta$. It is interesting to observe that for a smaller value of A, a large portion of the limit cycle is outside the strip $\mathcal{S}$. As A increases, the limit cycle Γ follows slowly the isocline near $x = \alpha$ (respectively, near $x = \beta$) till to the point of maximum (respectively, minimum). This maximum value is little below (respectively, above) $y = \rho^+(x)$ (respectively $y = \rho^-(x)$). Then the trajectory jumps fast (and moving almost horizontally) to meet the isocline again near $x = \beta$ (respectively, near $x = \alpha$). In this case, only a smaller portion of the limit cycle remains outside the horizontal strip.

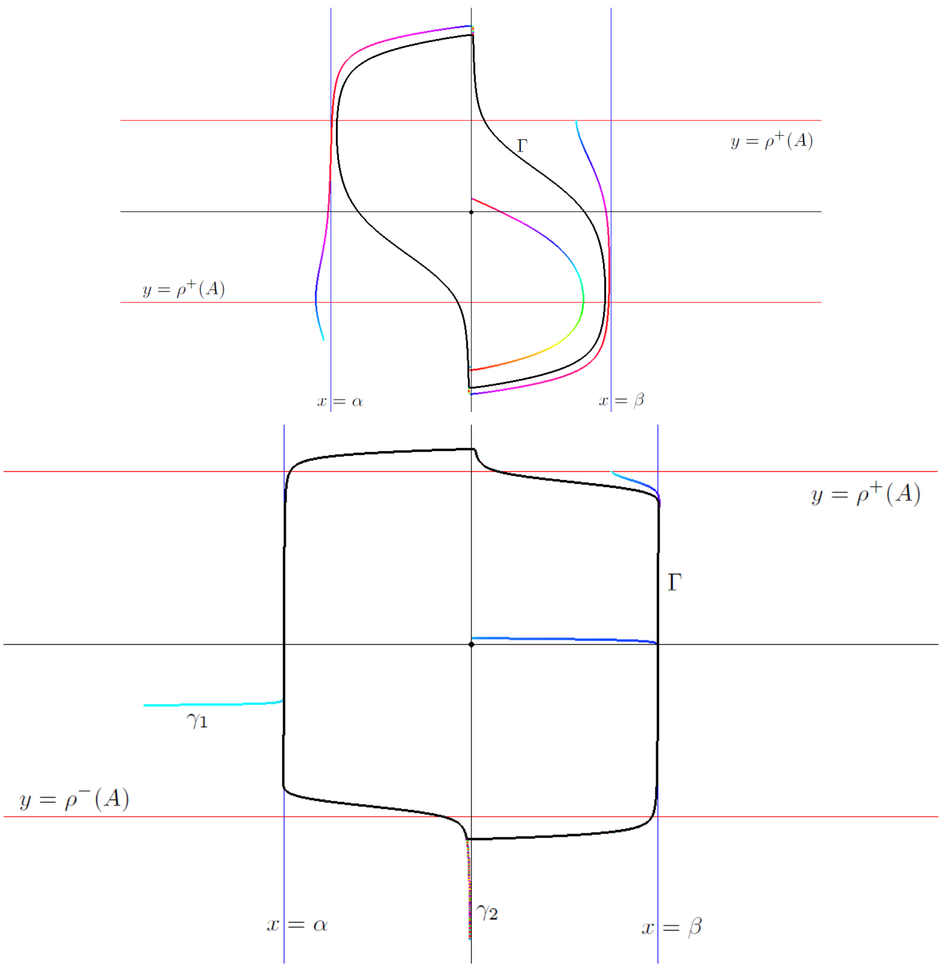

Figure 15. Existence of a limit cycle of system (2.24) crossing the strip $\mathcal{S}$ in the symmetric case $m = 4$ for $p = 2$. We have $A = 1/2$ in the upper panel and $A = 8$ in the lower panel. The phase-portraits are almost the same, from a qualitative point of view, except that for the larger value of A, the part of the limit cycle outside the strip is closer to the asymptotes $y = \rho^{\pm}(x)$.

The behavior illustrated in Figure 14 and Figure 15 fits completely with the content of Remark 2.9 where we have proved that enlarging m has the effect of reducing A (and therefore the size of the strip) and enlarging the constant λ. Thus is clear that for $m >> 1$, the crucial estimate (2.23) will fail (at least for $p = 2$).

3. **On the uniqueness of the limit cycle when A is large**

We continue our analysis of system (2.1). We have already proved that for $A > 0$ and large there exists at most a limit cycle which is contained in the strip

$$\mathcal{S} = \mathbb{R} \times (\rho^-(A), \rho^+(A)).$$

Throughout this paragraph, we keep the same hypotheses on F, g and R as in Section 2.3.

We are now interested in the study of the problem of the uniqueness of the limit cycle, for which the existence has been guaranteed in the previous paragraph. To this end, we confine the study of our equation to the strip $\mathcal{S}$. In this case, we can write system (2.1) as

$$\begin{cases} \dot{x} = (A - R(y)) \left(\frac{y}{A - R(y)} - \lambda F(x) \right) \\ (A - R(y))\dot{y} = -(A - R(y))g(x) \end{cases}$$

and then as

$$\begin{cases} \dot{x} = (A - R(y)) \left(\Psi_A(y) - \lambda F(x) \right) \\ \frac{d}{dt} W(y) = -(A - R(y))g(x), \end{cases}$$

where

$$W(y) = \int_0^y (A - R(s))\, ds.$$

The function W is a strictly increasing function defined on $(\rho^-(A), \rho^+(A))$ and having as range an open interval $(\sigma^-(A), \sigma^+(A))$. Then, setting $w = W(y)$ and $y = W^{-1}(w)$, we can pass to the equivalent system

$$\begin{cases} \dot{x} = (A - R(y)) \left(\Psi_A(W^{-1}(w)) - \lambda F(x) \right) \\ \dot{w} = -(A - R(y))g(x), \end{cases}$$

which, in turns, has the same orbits of the generalized Liénard system

$$\begin{cases} \dot{x} = h(w) - \lambda F(x) \\ \dot{w} = -g(x), \end{cases} \tag{3.1}$$

for

$$h(w) = \Psi_A(W^{-1}(w)).$$

By the above positions, it is straightforward to check that

$$h : (\sigma^-(A), \sigma^+(A)) \to \mathbb{R}$$

is a strictly increasing and smooth function with $h(0) = 0$ and such that

$$\lim_{w \to \sigma^-(A)^+} h(w) = -\infty, \qquad \lim_{w \to \sigma^+(A)^-} h(w) = +\infty.$$

By a theorem by Huang and Sun [21] and invoking also the remarks and corrections by Kooij and Sun [22], as well as by Zhang and Ping [54], we can conclude with the following.

Theorem 3.1. *There is at most one limit cycle contained in the strip $\mathcal{S}$ and crossing both the vertical lines $x = \alpha$ and $x = \beta$.*

For the classical Liénard systems, conditions guaranteeing that the limit cycles intersect both the vertical lines $x = \alpha$ and $x = \beta$ are widely studied in [4, 18, 19, 20, 39, 44, 49, 50]. In particular, if $F(x)$ is odd, this property holds.

As a conclusion, from Theorem 3.1 and Theorem 2.15, we obtain, as a corollary, Theorem 1.2 for equation (1.6), stated in the Introduction. Actually, we can prove a more general result for system (2.16), namely,

$$\dot{x} = y - \lambda(A - |y|^p)(x^3 - x), \qquad \dot{x} = y,$$

which provides further information with respect to Corollary (2.10).

Corollary 3.2. *Assume $1 < p \leq 2$. Then, the following conclusion holds.*

- *For every $\lambda > 0$ and $A > 0$, (2.16) has at least a stable limit cycle $\Gamma' = \Gamma'_{\lambda,A}$.*
- *There exists Λ^* such that for each $\lambda \in (0, \Lambda^*)$, there is $A^* = A^*(\lambda)$, such that for each $A \in (0, A^*)$ there is at least a second unstable limit cycle $\Gamma'' = \Gamma''_{\lambda,A}$, with Γ'' surrounding Γ'. We can take $\lambda^* = 1$ for $p = 2$.*
- *There exists $A^{**} = A^{**}(\lambda)$, such that for every $A > A^{**}$ there is a unique asymptotically stable limit cycle.*

4. The sublinear case

We have focused our attention in the preceding sections mainly to the superlinear case in which $R(y) \approx |y|^p$ with $p > 1$. The reason is that in this situation the most interesting dynamics occurs, as we have proven in [52]. In particular,for system (1.4) there are values of $\lambda > 0$ such that there are no (nontrivial) periodic orbits, as recalled in Theorem 1.1 (see also [11]). We consider now the sublinear case $0 < p < 1$.

For simplicity, we analyze the system with a symmetric van der Pol–Liénard function $F(x) = x(x^2 - 1)$, namely,

$$\begin{cases} \dot{x} = y - \lambda(A - |y|^p)(x^3 - x) \\ \dot{y} = -x. \end{cases} \tag{4.1}$$

First of all, we recall [52, Theorem 3] that we state as follows, using the inversion of time method applied in the proof of Theorem 2.2.

Theorem 4.1. *Assume* $0 < p < 1$, *then system*

$$\dot{x} = y + \lambda |y|^p (x^3 - x), \qquad \dot{y} = -x. \tag{4.2}$$

has a unique asymptotically unstable limit cycle, for every $\lambda > 0$.

This guarantees the existence of an unstable limit cycle to system (4.1) provided that $A > 0$ is sufficiently small as in Theorem 2.2, without restrictions on λ (but with A^* depending on λ). From now on, all the other results in Section 2 apply, in particular, Theorem 2.7 of the existence of a stable limit cycle for all $A > 0$, Theorem 2.12 on the existence of separatrices (for $A > 0$ and large) and Theorem 2.14 on the confinement of the limit cycles in the strip $\mathcal{S}$. Hence, we can then apply the uniqueness result of Theorem 3.1 and, finally, we obtain a version of Corollary 2.10 which reads as follows.

Theorem 4.2. *Assume* $0 < p < 1$. *Then, the following conclusion holds.*

- *For every* $\lambda > 0$ *and* $A > 0$, (2.16) *has at least a stable limit cycle* $\Gamma' = \Gamma'_{\lambda,A}$.
- *For every* $\lambda > 0$ *there exists* $A^* = A^*(\lambda)$, *such that for each* $A \in (0, A^*)$ *there is at least a second unstable limit cycle* $\Gamma'' = \Gamma''_{\lambda,A}$, *with* Γ'' *surrounding* Γ'.
- *There exists* $A^{**} = A^{**}(\lambda)$, *such that for every* $A > A^{**}$ *there is a unique asymptotically stable limit cycle.*

Figure 16 and Figure 17 are merely an illustration of the content of Theorem 4.2.

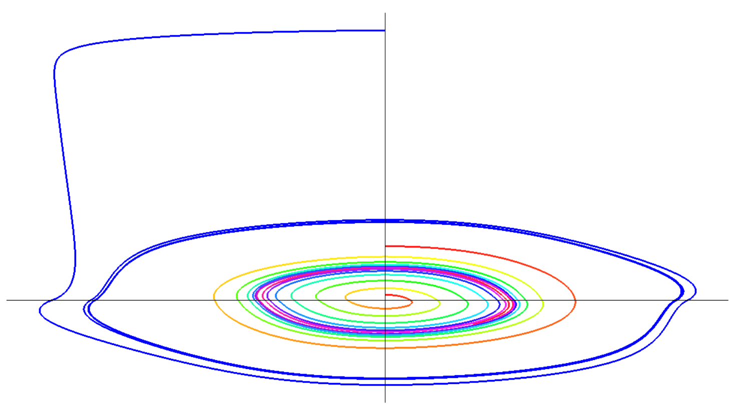

FIGURE 16. Example of two limit cycles for system (4.1) with $p = 1/2$, $\lambda = 1$, $A = 1/2$.

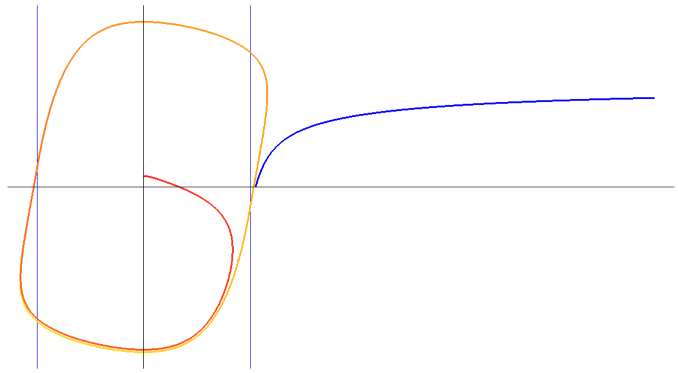

FIGURE 17. Example of a unique limit cycle for system (4.1) with $p = 1/2$, $\lambda = 1$, $A = 3$.

Notice that, as expected, the limit cycle cuts both lines $x = \alpha = -1$ and $x = \beta = 1$ (this because, due to the symmetry, inside the strip the energy increases). The limit cycle is contained in the strip $|y| < A^{1/p} = 9$ and therefore it is possible to apply the uniqueness result proved in Section 3. From the figure it is also evident the existence of a separatrix, which is crucial for the phenomenon of disappearance of the external unstable limit cycle.

The same results hold for $p = 1$ for the equation

$$\dot{x} = y - \lambda(A - |y|)(x^3 - x), \qquad \dot{y} = -x. \tag{4.3}$$

The only difference is that now [52, Theorem 1] has to be applied in place of [52, Theorem 3] and, accordingly, instead of Theorem 4.1 we have the following

Theorem 4.3. *The system*

$$\dot{x} = y + \lambda|y|(x^3 - x), \qquad \dot{y} = -x. \tag{4.4}$$

has a unique asymptotically unstable limit cycle, for every $\lambda \in (0, \Lambda^*)$, *with* $\Lambda^* = 3\sqrt{3}/2$ *and no limit cycles for* $\lambda > \Lambda^*$.

From this result and the theorems in Section 2 and Section 3, we can easily derive a version of Theorem 4.2 for system (4.3), also in line with Corollary 3.2.

5. Conclusions

As mentioned in the Introduction, this work continues the research initiated in [52] and in [11] dealing with a new class of planar systems of the form

$$\dot{x} = y - \lambda|y|^p F(x), \qquad y = -g(x) \tag{5.1}$$

($\lambda > 0$), where $g(x)$ and $F(x)$ have the typical shape coming from the study of the van der pol equation in the Liénard plane, namely $g(x)x > 0$ for $x \neq 0$, with $G(x) = \int_0^x g(s)\,ds \to +\infty$ as $x \to \pm\infty$ and $F(x)$ having three zeros $\alpha < 0 < \beta$ and such that $xF(x) > 0$ for $x \notin [\alpha, \beta]$ and $F(x)x < 0$ for $x \in (\alpha, \beta) \setminus \{0\}$. We also assume that $F(x)$ is increasing on $(-\infty, \alpha)$ and $(\beta, +\infty)$. This latter condition is crucial for the theorem on the uniqueness of the limit cycle and not needed in other existence results. It was proved in [52] that there is Λ^* such that for each $\lambda \in (0, \Lambda^*)$ (5.1) has at least a stable limit cycle, which is unique for $0 < p \leq 1$, while, $p > 1$ (the *superlinear case*) there are no limit cycles for $\lambda > \Lambda^*$. Moreover, no restriction on $\lambda > 0$ is needed in the *sublinear case* $0 < p < 1$. The same results provide the existence of an unstable limit cycle (for $\lambda \in (0, \Lambda^*)$) for the system

$$\dot{x} = y + \lambda |y|^p F(x), \qquad y = -g(x)$$

and, by a continuity argument, the existence of a second (small) limit cycle for

$$\dot{x} = y - \lambda(A - \lambda |y|^p) F(x), \qquad y = -g(x) \tag{5.2}$$

for A positive small, that is $A \in (0, A^*)$.

In this work we have proved that (5.2) and its generalization

$$\dot{x} = y - \lambda(A - \lambda R(y)) F(x), \qquad y = -g(x), \tag{5.3}$$

for $R(y)$ an even function increasing on $(0, +\infty)$ and such that $R(y) \to +\infty$ as $y \to \pm\infty$ (that we have studied in detail in Section 2 and Section 3) present a very rich dynamics, including the existence of at least two limit cycles when A is small, the permanence of only one stable limit cycle when A grows as well as the presence of unbounded separatrices and bounds for the regions containing the limit cycles. These latter properties allow to prove rigorously the uniqueness of a stable limit cycle when A is large. We also find a very delicate existence non-existence dichotomy depending on the symmetry relation $G(\alpha) = G(\beta)$ considered in [4] for the problem of uniqueness of limit cycles. The coexistence of all these properties in a relatively simple single system appears new in the literature and suggest the interest in a deep investigation of the equations of the class (5.3) and its subclass (5.2).

6. **Appendix: Hopf bifurcation from the origin**

Our aim here is to present a general result for a system of the form

$$\begin{cases} \dot{x} = y - (A - R(y)) F(x) \\ \dot{y} = -g(x) \end{cases} \tag{6.1}$$

which shows, under a minimal set of hypotheses on the nonlinearities, the occurrence of a Hopf bifurcation [3] of small asymptotically stable limit cycles γ_a departing from the origin and parameterized by $a > 0$ and small. For convenience, we have taken (1.7) with $\lambda = 1$, since the value of $\lambda > 0$ does not effect the result about the Hopf

[3] also called Poincaré-Andronov-Hopf bifurcation in [7] (See [32] for a brief historical account on the development of the subject).

bifurcation. Moreover, this also avoid confusion with the fact that in the proof below, we denote by λ an eigenvalue.

System (6.1) fits into the form of

$$\begin{cases} \dot{u} = X(u,v;\mu) \\ \dot{v} = Y(u,v;\mu) \end{cases} \tag{6.2}$$

with X, Y sufficiently smooth functions depending on the real parameter μ and such that

$$X(0,0;\mu) = Y(0,0;\mu) = 0, \quad \forall\, \mu.$$

Let $\lambda(\mu) = \alpha(\mu) \pm i\beta(\mu)$ be the eigenvalues of the Jacobian matrix

$$\begin{pmatrix} X_1(0,0;\mu) & X_2(0,0;\mu) \\ Y_1(0,0;\mu) & Y_2(0,0;\mu) \end{pmatrix}$$

at the origin (where the subscripts indicate the derivative with respect to the first or second variable). Following [14], we state an equivalent version of the Hopf bifurcation theorem for two-dimensional systems in a form that is suitable for the application to (6.1)

Theorem 6.1. *Assume that the following conditions hold.*

$(H1)$ $Y_1(0,0;0) = 1 = -X_2(0,0;0)$, *and* $X_1(0,0;0) = 0 = Y_2(0,0;0)$;
$(H2)$ $\frac{d\alpha(\mu)}{d\mu}|_{\mu=0} = d \neq 0$;
$(H3)$ $\rho = M + N \neq 0$, *with*

$$M := X_{111} + X_{122} + Y_{221} + Y_{222}$$
$$N := X_{12}(X_{11} + X_{22}) - Y_{12}(Y_{11} + Y_{22}) - X_{11}Y_{11} + X_{22}Y_{22}$$

and where all the derivatives are evaluated at $(0,0)$ *for* $\mu = 0$.

Then a unique curve of periodic solutions bifurcates from the equilibrium point for $\mu > 0$ *if* $\rho d < 0$ *and for* $\mu < 0$ *if* $\rho d > 0$. *In the former case the resulting limit cycle is stable and it is unstable in the latter case.*

Conditions $(H1)$ and $(H2)$ are quite natural and express a change of stability of the equilibrium point as μ crosses the value $\mu = 0$. We have expressed $(H1)$ in a simplified form with respect to [14]. However, one can always reduce to this case, under the more commonly used assumption $\alpha(0) = 0$ and $\beta(0) = 1$.

It is easy to find examples for which the first two conditions alone are not enough to produce a bifurcation of limit cycles from the fixed point. Condition $(H3)$ is written in several different manners in the texts exposing Hopf bifurcation theory, depending by the representation of the vector field (X, Y) in normal form using the Taylor expansion to the order three. It is a so-called "genericity condition" and it is related to the non-vanishing of the first Lyapunov coefficient [23] (or Liapunov number according to Perko [38]) The coefficient ρ (up to a multiplicative constant) is usually denoted by a in some books like [14] (or σ in [38] or $\ell_1(0)$ in [23]). We prefer to use a different symbol, as the coefficient a is already introduced in system (1.6) with a completely different meaning. The explicit form of condition $(H3)$ that we use here corresponds to [14, Formula (3.4.11)] for $\beta(0) = \omega = 1$. For a similar version of condition $(H3)$ with an application to planar systems, see also [8].

Now, the following result holds.

Theorem 6.2. *Assume that g, F, R are sufficiently smooth (at least of class C^3) in a neighborhood of the origin and satisfy the following local conditions:*

- $(A1)$ $g(0) = F(0) = 0$, $g'(0) > 0$ *and* $F'(0) \neq 0$;
- $(A2)$ $R(0) = R'(0) = 0$, $R''(0) > 0$.

Then there exists $\hat{A} > 0$ and a one-parameter family $(0, \hat{A}) \ni A \mapsto \gamma_A$ of asymptotically stable limit cycles of system (6.1), *bifurcating from the origin.*

Proof. We consider the system

$$\begin{cases} \dot{x} = y - (\mu - R(y))F(x) \\ \dot{y} = -g(x) \end{cases} \tag{6.3}$$

depending on the parameter $\mu \in \mathbb{R}$ (for the moment, also $\mu < 0$ is allowed). In applying Theorem 6.1 to equation (6.3) we set $u = x$ and $v := -y$, so that

$$X(u, v, \mu) := -v - \mu F(u) + R(-v)F(u), \qquad Y(u, v, \mu) := -g(u).$$

We also suppose, without loss of generality, that $g'(0) = 1$, because, due to the assumption $g'(0) = L > 0$, we can always reduce to the case $L = 1$, via an elementary rescaling. With these positions, the Jacobian matrix at the origin is given by

$$\begin{pmatrix} -\mu F'(0) & -1 \\ g'(0) = 1 & 0 \end{pmatrix}$$

with eigenvalues $\alpha(\mu) \pm i\beta(\mu)$, where

$$\alpha(\mu) = -\mu\frac{F'(0)}{2}, \quad \beta(\mu) = \sqrt{1 - \alpha(\mu)^2}, \quad \text{for } |\mu| < k_0 := \frac{2}{F'(0)}.$$

Conditions $(H1)$ and $(H2)$ are clearly satisfied, with $d = -\frac{F'(0)}{2} \neq 0$.

In order to check condition $(H3)$, we observe that $Y_2 \equiv 0$, as the second equation depends only on the first variable. Hence, using $R(0) = R'(0) = 0$, we have $M = R''(0)F'(0)$ and $N = 0$, so that

$$\rho d = -\frac{1}{2}R''(0)F'(0)^2 < 0.$$

Since $\rho d < 0$, Theorem 6.1 implies that a one-parameter family of asymptotically stable limit cycles of (6.3) bifurcates from the origin for $\mu > 0$ and small. Having proved the validity of our result for $\mu > 0$, the conclusion applies immediately to system (6.1) for $A = \mu$ positive and small. $\square$

We notice that Theorem 6.2 involves only conditions on the sign of the derivatives on the nonlinear terms. Therefore, it applies to system (2.1) with $\lambda > 0$, leading to the following.

Corollary 6.3. *Assume that $g, F, R : \mathbb{R} \to \mathbb{R}$ satisfy the same assumptions as in Theorem 6.2. Then for every $\lambda > 0$, there exists $\hat{A}(\lambda) > 0$ and a one-parameter family $(0, \hat{A}(\lambda)) \ni A \mapsto \gamma_A$ of asymptotically stable limit cycles of system* (2.1)*, bifurcating from the origin.*

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