Neural, Parallel, and Scientific Computations 33 (2025) 39-54

GENERALIZED LOGISTIC NEURAL NETWORKS AS POSITIVE LINEAR OPERATORS OVER INFINITE DOMAIN

GEORGE A. ANASTASSIOU

Department of Mathematical Sciences, University of Memphis, Melbourne,
Memphis, TN 38152, USA. ganastss@memphis.edu

ABSTRACT. Generalized Logistic neural network operators in infinite domain are interpreted as positive linear operators and related general theory applies to them. These operators are induced by a symmetrized density function deriving from the parametrized, deformed and symmetrized Generalized Logistic activation function. We are acting on the space of continuous and bounded functions on the real line to the reals. We study quantitatively the rate of convergence of these neural network operators to the unit operator. Our inequalities involve the modulus of continuity of the function under approximation or its derivative under initial conditions.

AMS (MOS) Subject Classification: 41A17, 41A25, 41A36

Keywords and phrases: Neural Network operators, Positive linear Operators, modulus of continuity, quantitative approximation to the unit, infinite domain,. generalized logistic activation function.

1. Introduction

The author studied extensively the quantitative approximation of positive linear operators to the unit since 1985, see for example [1]-[3], [6], which we use in this work. He originated from the quantitative weak convergence of finite positive measures to the unit Dirac measure, having as a method the geometric moment theory, see [2], and he produced best upper bounds, leading to attained (i.e. sharp Jackson type inequalities), e.g. see [1], [2]. These studies have been gone to all possible directions, univariate and multivariate, though in this work we stay only on the univariate approach over infinite domain.

Also, this author since 1997, he started the study of the quantitative convergence of neural network operators to the unit, and he has been written since then numerous of articles and books, e.g. see [3], [5], [6], are what we use here.

The wide range of neural network operators over $\mathbb{R}$ being treated by the author all along, are indeed by nature positive linear operators.

Received April 20, 2025 ISSN 1061-5369 © Dynamic Publishers, Inc.
https://doi.org/10.46719/npsc2025.33.03 www.dynamicpublishers.org .

Here, for the first time in the literature, neural network operators in infinite domain are treated as positive linear operators. Now the activation function comes from the generalized logistic function.

So, methods of positive linear operators in infinite domain apply here to our infinite summation defined neural network operators, producing new and interesting pointwise and uniform results.

Of great inspiration always have been [8], [9]. For classic studies on neural network we recommend also, [10]-[15].

For newer work on neural networks we also refer the reader to [16]-[25].

2. **Basics**

Here we follow [5], pp. 395-417.

Our activation function here to be used is the q-deformed and λ-parametrized function

$$\varphi_{q,\lambda}(x) = \frac{1}{1+qA^{-\lambda x}}, \quad x \in \mathbb{R}, \; q, \lambda > 0, \; A > 1. \tag{1}$$

This is the A-generalized logistic function.

For more read Chapter 16 of [5]: "Banach space valued ordinary and fractional neural network approximation based on q-deformed and λ-parametrized A-generalized logistic function".

This chapter motivates our current work.

The proposed "symmetrization technique" aims to use half data feed to our neural networks.

We will employ the following density function

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R}, \; q, \lambda > 0. \tag{2}$$

We have that

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \tag{3}$$

and

$$G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x), \; \forall \, x \in \mathbb{R}. \tag{4}$$

Adding (3) and (4) we obtain

$$G_{q,\lambda}(-x) + G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x), \; \forall \, x \in \mathbb{R}, \tag{5}$$

the key to this work.

So that

$$W(x) := \frac{G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x)}{2} \tag{6}$$

is an even function, symmetric with respect to the y-axis.

The global maximum of $G_{q,\lambda}$ is given by (16.18), p. 401 of [5] as

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right)=\frac{A^{\lambda}-1}{2\left(A^{\lambda}+1\right)}. \tag{7}$$

And, the global max of $G_{\frac{1}{q},\lambda}$ is

$$G_{\frac{1}{q},\lambda}\left(\frac{\log_A \frac{1}{q}}{\lambda}\right)=G_{\frac{1}{q},\lambda}\left(\frac{-\log_A q}{\lambda}\right)=\frac{A^{\lambda}-1}{2\left(A^{\lambda}+1\right)}, \tag{8}$$

both sharing the same maximum at symmetric points.

By Theorem 16.1, p. 401 of [5], we have that

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}\left(x-i\right)=1,\ \ \forall\ x\in\mathbb{R},\ \lambda,q>0,\ \ A>1, \tag{9}$$

and

$$\sum_{i=-\infty}^{\infty} G_{\frac{1}{q},\lambda}\left(x-i\right)=1,\ \ \forall\ x\in\mathbb{R},\ \lambda,q>0,\ \ A>1. \tag{10}$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} W\left(x-i\right)=1,\ \ \forall\ x\in\mathbb{R}. \tag{11}$$

By Theorem 16.2, p. 402 of [4], we have that

$$\int_{-\infty}^{\infty} G_{q,\lambda}\left(x\right)dx=1,\quad \lambda,q>0,\ \ A>1, \tag{12}$$

similarly it holds

$$\int_{-\infty}^{\infty} G_{\frac{1}{q},\lambda}\left(x\right)dx=1, \tag{13}$$

so that

$$\int_{-\infty}^{\infty} W\left(x\right)dx=1, \tag{14}$$

therefore $W\left(x\right)$ is a density function.

By Theorem 16.3, p. 402 of [5], we have:

Let $0<\beta<1$, and $N\in\mathbb{N}$ with $N^{1-\beta}>2$. Then

$$\sum_{\substack{k=-\infty \\ :\left|Nx-k\right|\geq N^{1-\beta}}}^{\infty} G_{q,\lambda}\left(Nx-k\right)<2\max\left\{q,\frac{1}{q}\right\}\frac{1}{A^{\lambda\left(N^{1-\beta}-2\right)}}=\gamma A^{-\lambda\left(N^{1-\beta}-2\right)}, \tag{15}$$

where $\lambda,q>0,\ A>1$; $\gamma:=2\max\left\{q,\frac{1}{q}\right\}$.

Similarly, we get that

$$\sum_{\substack{k=-\infty \\ : |Nx-k| \geq N^{1-\beta}}}^{\infty} G_{\frac{1}{q},\lambda}(Nx-k) < \gamma A^{-\lambda\left(N^{1-\beta}-2\right)}. \tag{16}$$

Consequently we obtain that

$$\sum_{\substack{k=-\infty \\ : |Nx-k| \geq N^{1-\beta}}}^{\infty} W(Nx-k) < \gamma A^{-\lambda\left(N^{1-\beta}-2\right)}, \tag{17}$$

where $\gamma := 2\max\left\{q, \frac{1}{q}\right\}$.

Here $\lceil\cdot\rceil$ denotes the ceiling of the number, and $\lfloor\cdot\rfloor$ its integral part.

We make

Remark 2.1. Let $x \in [1, \infty)$, and we apply the mean value theorem. We have that

$$G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(q-1)\right)$$

$$\frac{1}{2}\varphi'_{q,\lambda}(\xi)\, 2 = \varphi'_{q,\lambda}(\xi) = \frac{q\lambda \ln A}{\left(1+qA^{-\lambda\xi}\right)^2 A^{\lambda\xi}}, \tag{18}$$

where $0 \leq x-1 < \xi < x+1$.

We obtain that

$$G_{q,\lambda}(x) < (q\lambda \ln A)\, A^{-\lambda\xi} < (q\lambda \ln A)\, A^{-\lambda(x-1)}, \quad x \geq 1. \tag{19}$$

Similarly it holds

$$G_{\frac{1}{q},\lambda}(x) < \left(\frac{1}{q}\lambda \ln A\right) A^{-\lambda(x-1)}, \quad x \geq 1. \tag{20}$$

And, fimally it holds

$$W(x) < \frac{1}{2}\left(q + \frac{1}{q}\right)(\lambda \ln A)\, A^{-\lambda(x-1)}, \quad x \geq 1, \tag{21}$$

where $q, \lambda > 0$, $A > 1$.

We also make

Remark 2.2. We would like to estimate ($N, m \in \mathbb{N}$):

$$\sum_{\substack{k=-\infty \\ : |Nx-k| \geq N^{1-\beta}}}^{\infty} W(|Nx-k|)\, |Nx-k|^m \overset{(21)}{\leq}$$

$$\frac{1}{2}\left(q+\frac{1}{q}\right)(\lambda \ln A) \sum_{\begin{cases} k=-\infty \\ : |Nx-k| \geq N^{1-\beta} \end{cases}}^{\infty} |Nx-k|^m A^{-\lambda(|Nx-k|-1)} = \tag{22}$$

$$\frac{1}{2}\left(q+\frac{1}{q}\right)(\lambda \ln A) \sum_{\begin{cases} k=-\infty \\ : |Nx-k| \geq N^{1-\beta} \end{cases}}^{\infty} |Nx-k|^m e^{-\lambda(\ln A)(|Nx-k|-1)} =$$

$$\frac{1}{2}\left(q+\frac{1}{q}\right)\lambda(\ln A) A^{\lambda} \sum_{\begin{cases} k=-\infty \\ : |Nx-k| \geq N^{1-\beta} \end{cases}}^{\infty} |Nx-k|^m e^{-\lambda(\ln A)|Nx-k|} =: (*).$$

For convenience set $\mu := \lambda \ln A > 0$.

Notice that

$$e^{\frac{\mu|Nx-k|}{2}} = \sum_{\theta=0}^{\infty} \frac{\left(\frac{\mu|Nx-k|}{2}\right)^{\theta}}{\theta!} \geq \left(\frac{\mu|Nx-k|}{2}\right)^m \frac{1}{m!}. \tag{23}$$

Therefore we have

$$\left(\frac{\mu|Nx-k|}{2}\right)^m \leq m! e^{\frac{\mu|Nx-k|}{2}}, \quad \text{or} \tag{24}$$

$$(\mu|Nx-k|)^m \leq 2^m m! e^{\frac{\mu|Nx-k|}{2}}, \text{ or} \tag{25}$$

$$|Nx-k|^m \leq \frac{2^m}{\mu^m} m! e^{\frac{\mu|Nx-k|}{2}}. \tag{26}$$

Hence it holds

$$(*) \leq \frac{1}{2}\left(q+\frac{1}{q}\right)\lambda(\ln A) A^{\lambda}$$

$$\sum_{\begin{cases} k=-\infty \\ : |Nx-k| \geq N^{1-\beta} \end{cases}}^{\infty} \frac{2^m}{\lambda^m (\ln A)^m} m! e^{\frac{\lambda \ln A|Nx-k|}{2}} e^{-\lambda \ln A|Nx-k|} =$$

$$\frac{1}{2}\left(q+\frac{1}{q}\right)\lambda(\ln A) A^{\lambda} \frac{2^m}{\lambda^m (\ln A)^m} m! \sum_{\begin{cases} k=-\infty \\ : |Nx-k| \geq N^{1-\beta} \end{cases}}^{\infty} e^{-\frac{\lambda \ln A|Nx-k|}{2}} = \tag{27}$$

$$\frac{1}{2}\left(q+\frac{1}{q}\right)\frac{A^{\lambda}2^{m}m!}{\lambda^{m-1}(\ln A)^{m-1}}\left(\sum_{\begin{cases} k=-\infty \\ :|Nx-k|\geq N^{1-\beta}\end{cases}}^{\infty} e^{-\frac{\lambda\ln A}{2}|Nx-k|}\right)\leq$$

$$\left(q+\frac{1}{q}\right)\frac{A^{\lambda}2^{m}m!}{\lambda^{m-1}(\ln A)^{m-1}}\left(\int_{N^{1-\beta}-1}^{\infty} e^{-\frac{\lambda\ln A}{2}x}dx\right)=$$

$$\frac{2}{\lambda\ln A}\left(q+\frac{1}{q}\right)\frac{A^{\lambda}2^{m}m!}{\lambda^{m-1}(\ln A)^{m-1}}\left(\int_{N^{1-\beta}-1}^{\infty} e^{-\frac{\lambda\ln A}{2}x}d\left(\frac{\lambda\ln A}{2}\right)x\right)= \tag{28}$$

$$\frac{\left(q+\frac{1}{q}\right)A^{\lambda}2^{m+1}m!}{\lambda^{m}(\ln A)^{m}}\left(\int_{N^{1-\beta}-1}^{\infty} e^{-y}dy\right)=$$

$$\frac{\left(q+\frac{1}{q}\right)A^{\lambda}2^{m+1}m!}{\lambda^{m}(\ln A)^{m}}\left(-e^{-y}|_{N^{1-\beta}-1}^{\infty}\right)=$$

$$\frac{\left(q+\frac{1}{q}\right)A^{\lambda}2^{m+1}m!}{\lambda^{m}(\ln A)^{m}}\left(e^{-y}|_{\infty}^{N^{1-\beta}-1}\right)= \tag{29}$$

$$\frac{\left(q+\frac{1}{q}\right)A^{\lambda}2^{m+1}m!}{\lambda^{m}(\ln A)^{m}}\left(e^{-\frac{\lambda\ln A}{2}x}|_{\infty}^{N^{1-\beta}-1}\right)=$$

$$\frac{\left(q+\frac{1}{q}\right)A^{\lambda}2^{m+1}m!}{\lambda^{m}(\ln A)^{m}}\left(e^{-\frac{\lambda\ln A}{2}\left(N^{1-\beta}-1\right)}\right).$$

Thus, we have proved:

$$\sum_{\begin{cases} k=-\infty \\ :|Nx-k|\geq N^{1-\beta}\end{cases}}^{\infty} W(|Nx-k|)|Nx-k|^{m}\leq$$

$$\frac{\left(q+\frac{1}{q}\right)2^{m+1}m!}{\lambda^{m}(\ln A)^{m}}A^{-\frac{\lambda}{2}\left(N^{1-\beta}-3\right)}, \tag{30}$$

where $m,N\in\mathbb{N}$; $q,\lambda>0$, $A>1$, $0<\beta<1$.

We also make

Remark 2.3. We also estimate ($\gamma:=2\max\left\{q,\frac{1}{q}\right\}$)

$$\sum_{\begin{cases} k=-\infty \\ :|Nx-k|\geq N^{1-\beta}\end{cases}}^{\infty} W(|Nx-k|)(1+|Nx-k|)^{m}\leq$$

$$2^{m-1} \sum_{\substack{k=-\infty \\ : |Nx-k| \geq N^{1-\beta}}}^{\infty} W(|Nx-k|)(1+|Nx-k|^m) = \tag{31}$$

$$2^{m-1} \left\{ \sum_{\substack{k=-\infty \\ : |Nx-k| \geq N^{1-\beta}}}^{\infty} W(|Nx-k|) + \sum_{\substack{k=-\infty \\ : |Nx-k| \geq N^{1-\beta}}}^{\infty} W(|Nx-k|)|Nx-k|^m \right\} \overset{\text{(by (17), (30))}}{\leq}$$

$$2^{m-1} \left\{ \gamma A^{-\lambda\left(N^{1-\beta}-2\right)} + \left(\frac{\left(q+\frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} \right) A^{-\frac{\lambda}{2}\left(N^{1-\beta}-3\right)} \right\}.$$

We have proved that

$$\sum_{\substack{k=-\infty \\ : |Nx-k| \geq N^{1-\beta}}}^{\infty} W(|Nx-k|)(1+|Nx-k|)^m \leq$$

$$2^{m-1} \left\{ \gamma A^{-\lambda\left(N^{1-\beta}-2\right)} + \left(\frac{\left(q+\frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} \right) A^{-\frac{\lambda}{2}\left(N^{1-\beta}-3\right)} \right\}, \tag{32}$$

where $m, N \in \mathbb{N}$; $q, \lambda > 0$, $A > 1$, $0 < \beta < 1$, $\gamma = 2 \max \left\{ q, \frac{1}{q} \right\}$.

We need

Definition 2.4. In this article we study the smooth approximation properties of the following interpolation neural network operators acting on $f \in C_B(\mathbb{R})$ (continuous and bounded functions):

(i) the basic ones

$$B_N(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{N}\right) W(Nx-k), \quad \forall\, x \in \mathbb{R},\ N \in \mathbb{N}, \tag{33}$$

(ii) the Kantorovich type operators

$$C_N(f,x) := \sum_{k=-\infty}^{\infty} \left(N \int_{\frac{k}{N}}^{\frac{k+1}{N}} f(t)\,dt \right) W(Nx-k), \quad \forall\, x \in \mathbb{R},\ N \in \mathbb{N}, \tag{34}$$

(iii) let $\theta \in \mathbb{N}$, $w_r \geq 0$, $\sum_{r=0}^{\theta} w_r = 1$, $k \in \mathbb{Z}$, and

$$\delta_{Nk}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{N} + \frac{r}{N\theta} \right), \tag{35}$$

we consider also the quadrature type operators

$$D_N(f,x) := \sum_{k=-\infty}^{\infty} \delta_{Nk}(f)\, W(Nx-k), \quad \forall\, x \in \mathbb{R},\ N \in \mathbb{N}. \tag{36}$$

We will be using the first modulus of continuity:

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in \mathbb{R}: \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{37}$$

where $f \in C(\mathbb{R})$ which is bounded and or uniformly continuous.

The space $C_u(\mathbb{R})$ are the uniformly continuous functions on $\mathbb{R}$, and $C_{uB}(\mathbb{R})$ the uniformly and bounded functions on $\mathbb{R}$.

We notice that $B_N(1) = C_N(1) = D_N(1) = 1$, and B_N, C_N, D_N are positive linear operators, and they map $C_B(\mathbb{R})$ into $C_B(\mathbb{R})$; for the last see (6) and [5], Ch. 15, Theorem 15.15, p. 391.

3. **Auxilliary Results**

We need the following important results.

Lemma 3.1. *Let here* $x \in \mathbb{R}$, $j \in \mathbb{N}$; $0 < \beta < 1$, $A > 1$, $q, \lambda > 0$; $N \in \mathbb{N}$ *with* $N^{1-\beta} > 2$. *Then*

$$0 < B_N\left(|\cdot - x|^j \right)(x) \leq$$

$$\frac{1}{N^{\beta j}} + \frac{1}{N^j} \left(\frac{\left(q + \frac{1}{q} \right) 2^{j+1} j!}{\lambda^j (\ln A)^j} \right) A^{-\frac{\lambda}{2}\left(N^{1-\beta}-3\right)} =: \psi_1(N) < +\infty, \tag{38}$$

and

$$0 < \left\| B_N\left(|\cdot - x|^j \right)(x) \right\|_\infty \leq \psi_1(N) \to 0, \text{ as } N \to \infty. \tag{39}$$

Proof. We have that

$$B_N\left(|\cdot - x|^j \right)(x) = \sum_{k=-\infty}^{\infty} \left| \frac{k}{N} - x \right|^j W(Nx-k) =$$

$$\sum_{\substack{k=-\infty \\ : \left|x-\frac{k}{N}\right| < \frac{1}{N^{\beta}}}}^{\infty} \left|\frac{k}{N}-x\right|^{j} W(Nx-k) +$$

(40) $$\sum_{\substack{k=-\infty \\ : \left|x-\frac{k}{N}\right| \geq \frac{1}{N^{\beta}}}}^{\infty} \left|\frac{k}{N}-x\right|^{j} W(Nx-k) \overset{(11)}{\leq}$$

$$\frac{1}{N^{\beta j}} + \frac{1}{N^{j}} \sum_{\substack{k=-\infty \\ : |Nx-k| \geq N^{1-\beta}}}^{\infty} |Nx-k|^{j} W(|Nx-k|) \overset{(30)}{\leq}$$

$$\frac{1}{N^{\beta j}} + \frac{1}{N^{j}} \left(\frac{\left(q+\frac{1}{q}\right) 2^{j+1} j!}{\lambda^{j} (\ln A)^{j}} \right) A^{-\frac{\lambda}{2}\left(N^{1-\beta}-3\right)} < +\infty.$$

Lemma 3.2. *All as in Lemma 3.1. Then*

$$0 < C_N\left(|\cdot - x|^{j}\right)(x) \leq \left(\frac{1}{N}+\frac{1}{N^{\beta}}\right)^{j} +$$

(41) $$\frac{2^{j-1}}{N^{j}} \left[\gamma A^{-\lambda\left(N^{1-\beta}-2\right)} + \left(\frac{\left(q+\frac{1}{q}\right) 2^{j+1} j!}{\lambda^{j} (\ln A)^{j}} \right) A^{-\frac{\lambda}{2}\left(N^{1-\beta}-3\right)} \right] =: \psi_2(N) < +\infty,$$

and

(42) $$0 < \left\| C_N\left(|\cdot - x|^{j}\right)(x) \right\|_{\infty} \leq \psi_2(N) \to 0, \text{ as } N \to \infty.$$

Proof. We have that

$$C_N\left(|\cdot - x|^{j}\right)(x) = \sum_{k=-\infty}^{\infty} W(Nx-k) \left(N \int_0^{\frac{1}{N}} \left| t + \frac{k}{N} - x \right|^{j} dt \right) \leq$$

$$\sum_{k=-\infty}^{\infty} W(Nx-k) \left(N \int_0^{\frac{1}{N}} \left(\left| \frac{k}{N} - x \right| + |t| \right)^{j} dt \right) \leq$$

$$\sum_{k=-\infty}^{\infty} W(Nx-k) \left(\left| \frac{k}{N} - x \right| + \frac{1}{N} \right)^{j} =$$

$$\sum_{\substack{k=-\infty \\ : \left|\frac{k}{N}-x\right| < \frac{1}{N^{\beta}}}}^{\infty} W(Nx-k) \left(\left| \frac{k}{N} - x \right| + \frac{1}{N} \right)^{j} +$$

$$\sum_{\substack{k=-\infty \\ : \left|\frac{k}{N}-x\right| \geq \frac{1}{N^{\beta}}}}^{\infty} W(Nx-k)\left(\left|\frac{k}{N}-x\right|+\frac{1}{N}\right)^{j} \leq \tag{43}$$

$$\left(\frac{1}{N^{\beta}}+\frac{1}{N}\right)^{j}+\frac{1}{N^{j}} \sum_{\substack{k=-\infty \\ : |Nx-k| \geq N^{1-\beta}}}^{\infty} W(|Nx-k|)(|Nx-k|+1)^{j} \overset{(32)}{\leq}$$

(44)

$$\left(\frac{1}{N^{\beta}}+\frac{1}{N}\right)^{j}+\frac{2^{j-1}}{N^{j}}\left\{\gamma A^{-\lambda\left(N^{1-\beta}-2\right)}+\left(\frac{\left(q+\frac{1}{q}\right) 2^{j+1} j!}{\lambda^{j}(\ln A)^{j}}\right) A^{-\frac{\lambda}{2}\left(N^{1-\beta}-3\right)}\right\}<+\infty .$$

Lemma 3.3. *All as in Lemma 3.1. Then*

$$0<D_{N}\left(|\cdot-x|^{j}\right)(x) \leq\left(\frac{1}{N}+\frac{1}{N^{\beta}}\right)^{j}+$$

$$\frac{2^{j-1}}{N^{j}}\left\{\gamma A^{-\lambda\left(N^{1-\beta}-2\right)}+\left(\frac{\left(q+\frac{1}{q}\right) 2^{j+1} j!}{\lambda^{j}(\ln A)^{j}}\right) A^{-\frac{\lambda}{2}\left(N^{1-\beta}-3\right)}\right\}=\psi_{2}(N)<+\infty, \tag{45}$$

and

$$0<\left\|D_{N}\left(|\cdot-x|^{j}\right)(x)\right\|_{\infty} \leq \psi_{2}(N) \rightarrow 0, \text{ as } N \rightarrow \infty. \tag{46}$$

Proof. As similar to Lemma 3.2 is omitted.

We need the following Hölder's type inequality for positive linear operators.

Theorem 3.4. *Let H be a positive linear operator from $C(\mathbb{R})$ into $C_{B}(\mathbb{R})$, and $f, g \in C(\mathbb{R})$, furthermore let $p, q>1: \frac{1}{p}+\frac{1}{q}=1$. Assume that $H\left(\left(|f(\cdot)|^{p}\right)\right)\left(s_{*}\right)$, $H\left(\left(|g(\cdot)|^{q}\right)\right)\left(s_{*}\right)>0$ for some $s_{*} \in \mathbb{R}$. Then*

$$H(|f(\cdot) g(\cdot)|)\left(s_{*}\right) \leq\left(H\left(\left(\left|f(\cdot)^{p}\right|\right)\right)\left(s_{*}\right)\right)^{\frac{1}{p}}\left(H\left(\left(\left|g(\cdot)^{q}\right|\right)\right)\left(s_{*}\right)\right)^{\frac{1}{q}} . \tag{47}$$

Proof. Let $a, b \geq 0$, $p.q>1: \frac{1}{p}+\frac{1}{q}=1$. The Young's inequality says

$$a b \leq \frac{a^{p}}{p}+\frac{b^{q}}{q}. \tag{48}$$

Then

$$\frac{|f(s)|}{\left(H\left(\left(\left|f(\cdot)^{p}\right|\right)\right)\left(s_{*}\right)\right)^{\frac{1}{p}}} \cdot \frac{|g(s)|}{\left(H\left(\left(\left|g(\cdot)^{q}\right|\right)\right)\left(s_{*}\right)\right)^{\frac{1}{q}}} \leq$$

$$\frac{|f(s)|^{p}}{p\left(H\left(\left(\left|f(\cdot)^{p}\right|\right)\right)\left(s_{*}\right)\right)}+\frac{|g(s)|^{q}}{q\left(H\left(\left(\left|g(\cdot)^{q}\right|\right)\right)\left(s_{*}\right)\right)}, \quad \forall\ s \in \mathbb{R}. \tag{49}$$

Hence it holds

$$\frac{H(|f(\cdot)||g(\cdot)|)\left(s_{*}\right)}{\left(H\left(\left(\left|f(\cdot)^{p}\right|\right)\right)\left(s_{*}\right)\right)^{\frac{1}{p}}\left(H\left(\left(\left|g(\cdot)^{q}\right|\right)\right)\left(s_{*}\right)\right)^{\frac{1}{q}}} \leq$$

$$\frac{H\left(\left(\left|f\left(\cdot\right)^{p}\right|\right)\right)\left(s_{*}\right)}{p\left(H\left(\left(\left|f\left(\cdot\right)^{p}\right|\right)\right)\left(s_{*}\right)\right)}+\frac{H\left(\left(\left|g\left(\cdot\right)^{q}\right|\right)\right)\left(s_{*}\right)}{q\left(H\left(\left(\left|g\left(\cdot\right)^{q}\right|\right)\right)\left(s_{*}\right)\right)}=\frac{1}{p}+\frac{1}{q}=1, \tag{50}$$

for $s_* \in \mathbb{R}$, proving the claim.

4. **Main Results**

We present the following general approximation result.

Theorem 4.1. *Let $n \in \mathbb{N}$ and $f, f^{(n)} \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. Consider H_N a sequence of positive linear operators from $C_B(\mathbb{R})$ into itself, $N \in \mathbb{N}$, such that $H_N(1) = 1$. Assume $H_N\left(|\cdot - x|^{n+1}\right)(x) > 0$, and $f^{(i)}(x) = 0$, for $i = 1, ..., n$.*

Then

$$|H_N(f)(x) - f(x)| \leq \omega_1\left(f^{(n)}, \left(\left(H_N\left(|\cdot - x|^{n+1}\right)\right)(x)\right)^{\frac{1}{n+1}}\right)$$

$$\left[H_N\left(|\cdot - x|^{n}\right)(x) + \frac{\left(H_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}}{(n+1)}\right] < +\infty, \quad \forall\ N \in \mathbb{N}. \tag{51}$$

Given that $\lim_{N\to+\infty} H_N\left(|\cdot - x|^{n+1}\right)(x) = 0$, then $\lim_{N\to+\infty} H_N(f)(x) = f(x)$.

If $n = 1$, we derive

$$|H_N(f)(x) - f(x)| \leq \omega_1\left(f', \left(\left(H_N\left((\cdot - x)^{2}\right)\right)(x)\right)^{\frac{1}{2}}\right)$$

$$\left[H_N\left(|\cdot - x|\right)(x) + \frac{\left(H_N\left((\cdot - x)^{2}\right)(x)\right)^{\frac{1}{2}}}{2}\right] < +\infty, \quad \forall\ N \in \mathbb{N}. \tag{52}$$

Given that $\lim_{N\to+\infty} H_N\left((\cdot - x)^{2}\right)(x) = 0$, then $\lim_{N\to+\infty} H_N(f)(x) = f(x)$.

Proof. Let $x, y \in \mathbb{R}$, by Taylor's formula we have

$$f(y) = \sum_{i=0}^{n} f^{(i)}(x)\frac{(y-x)^i}{i!} + \frac{1}{(n-1)!}\int_x^y (y-t)^{n-1}\left(f^{(n)}(t) - f^{(n)}(x)\right)dt. \tag{53}$$

We call the remainder in (53) as

$$R_n^*(x,y) = \frac{1}{(n-1)!}\int_x^y (y-t)^{n-1}\left(f^{(n)}(t) - f^{(n)}(x)\right)dt, \quad \forall\ x, y \in \mathbb{R}. \tag{54}$$

By [2], p. 217, and [4], p. 194, Chapter 7, (7.27) there, we get

$$|R_n^*(x,y)| \leq \frac{\omega_1\left(f^{(n)}, \delta\right)}{n!}|x-y|^n\left(1 + \frac{|x-y|}{(n+1)\delta}\right), \quad \forall\ x, y \in \mathbb{R},\ \delta > 0. \tag{55}$$

We may write (55) as

$$|R_n^*(x,y)| \leq \frac{\omega_1\left(f^{(n)}, \delta\right)}{n!}\left[|x-y|^n + \frac{|x-y|^{n+1}}{(n+1)\delta}\right], \quad \forall\ x, y \in \mathbb{R},\ \delta > 0. \tag{56}$$

That is

$$\left|f(y)-\sum_{i=0}^{n} f^{(i)}(x)\frac{(y-x)^i}{i!}\right| \leq \frac{\omega_1\left(f^{(n)},\delta\right)}{n!}\left[|x-y|^n+\frac{|x-y|^{n+1}}{(n+1)\delta}\right], \tag{57}$$

$\forall\ x,y\in\mathbb{R}$, $\delta>0$.

Furthermore it holds

$$|f(y)-f(x)| \leq \sum_{i=0}^{n}\left|f^{(i)}(x)\right|\frac{|y-x|^i}{i!}+\frac{\omega_1\left(f^{(n)},\delta\right)}{n!}\left[|x-y|^n+\frac{|x-y|^{n+1}}{(n+1)\delta}\right], \tag{58}$$

$\forall\ x,y\in\mathbb{R}$, $\delta>0$.

Here $f^{(i)}(x)=0$, for $i=1,...,n$; and for a specific $x\in\mathbb{R}$, we get

$$|f(y)-f(x)| \leq \frac{\omega_1\left(f^{(n)},\delta\right)}{n!}\left[|x-y|^n+\frac{|x-y|^{n+1}}{(n+1)\delta}\right],\quad \forall y\in\mathbb{R},\ \delta>0. \tag{59}$$

In case of $n=1$, we derive

$$|f(y)-f(x)| \leq |f'(x)|\,|y-x|+\omega_1(f',\delta)\left[|x-y|+\frac{(x-y)^2}{2\delta}\right], \tag{60}$$

$\forall\ x,y\in\mathbb{R}$, $\delta>0$.

If $n=1$ and $f'(x)=0$, for a specific $x\in\mathbb{R}$, we get that

$$|f(y)-f(x)| \leq \omega_1(f',\delta)\left[|x-y|+\frac{(x-y)^2}{2\delta}\right],\quad \forall y\in\mathbb{R},\ \delta>0. \tag{61}$$

Let $H_N : C_B(\mathbb{R}) \to C_B(\mathbb{R})$ a sequence of positive linear operators, $N\in\mathbb{N}$.

Let $f,g\in C_B(\mathbb{R})$, we have

$$f=f-g+g\leq |f-g|+g,$$

hence

$$H_N(f)\leq H_N(|f-g|)+H_N(g), \tag{62}$$

and

$$H_N(f)-H_N(g)\leq H_N(|f-g|). \tag{63}$$

Similalrly, it holds

$$H_N(g)-H_N(f)\leq H_N(|f-g|). \tag{64}$$

That is

$$|H_N(f)-H_N(g)|\leq H_N(|f-g|). \tag{65}$$

Next, we assume that $H_N(1)=1$, $\forall\ N\in\mathbb{N}$. Then

$$|H_N(f)(x)-f(x)|=|H_N(f)(x)-f(x)\,H_N(1)(x)|=$$

$$|H_N(f)(x) - H_N(f(x))(x)| = |H_N(f(\cdot) - f(x))(x)| \overset{(65)}{\leq}$$

$$H_N(|f(\cdot) - f(x)|)(x) \overset{(59)}{\leq} \tag{66}$$

$$\frac{\omega_1\left(f^{(n)}, \delta\right)}{n!}\left[H_N\left(|\cdot - x|^n\right)(x) + \frac{H_N\left(|\cdot - x|^{n+1}\right)(x)}{(n+1)\,\delta}\right],$$

$\forall\, x \in \mathbb{R}$, $\delta > 0$.

Here we assume and choose

$$\delta = \left(H_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{1}{n+1}} > 0. \tag{67}$$

Therefore we get

$$|H_N(f)(x) - f(x)| \leq \omega_1\left(f^{(n)}, \left(H_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right)$$

$$\left[H_N\left(|\cdot - x|^n\right)(x) + \frac{\left(H_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}}{(n+1)}\right], \tag{68}$$

$\forall\, N \in \mathbb{N}$.

If $n = 1$, we find that

$$|H_N(f)(x) - f(x)| \leq \omega_1\left(f', \left(H_N\left((\cdot - x)^2\right)(x)\right)^{\frac{1}{2}}\right)$$

$$\left[H_N(|\cdot - x|)(x) + \frac{\left(H_N\left((\cdot - x)^2\right)(x)\right)^{\frac{1}{2}}}{2}\right], \quad \forall\, N \in \mathbb{N}. \tag{69}$$

Using (67) and Theorem 3.4, for $g = 1$ and H_N such that $H_N(1) = 1$, we obtain

$$H_N\left(|\cdot - x|^n\right)(x) \leq \left(H_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}. \tag{70}$$

In case of $n = 1$, we derive

$$H_N(|\cdot - x|)(x) \leq \sqrt{\left(H_N\left((\cdot - x)^2\right)(x)\right)}. \tag{71}$$

An application of Theorem 4.1 follows.

Call θ_N any of the operators B_N, C_N, D_N, $N \in \mathbb{N}$.

Theorem 4.2. *Let $n \in \mathbb{N}$ and $f, f^{(n)} \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. Assume $f^{(i)}(x) = 0$, $i = 1, ..., n$. Then*

$$|\theta_N(f)(x) - f(x)| \leq \omega_1\left(f^{(n)}, \left(\left(\theta_N\left(|\cdot - x|^{n+1}\right)\right)(x)\right)^{\frac{1}{n+1}}\right)$$

$$\left[\theta_N\left(|\cdot - x|^n\right)(x) + \frac{\left(\theta_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}}{(n+1)}\right] < +\infty, \quad \forall\, N \in \mathbb{N}. \tag{72}$$

Here $\lim_{N \to +\infty} \theta_N(f)(x) = f(x)$.

If $n=1$, we obtain

$$\left|\theta_N(f)(x)-f(x)\right| \leq \omega_1\left(f',\left(\left(\theta_N\left((\cdot-x)^2\right)\right)(x)\right)^{\frac{1}{2}}\right)$$

$$\left[\theta_N(|\cdot-x|)(x)+\frac{\left(\theta_N\left((\cdot-x)^2\right)(x)\right)^{\frac{1}{2}}}{2}\right]<+\infty, \quad \forall\ N \in \mathbb{N}. \tag{73}$$

Again it holds $\lim_{N\to+\infty} \theta_N(f)(x)=f(x)$.

Proof. By Theorem 4.1 and Lemmas 3.1-3.3.

We mention the following result.

Theorem 4.3. *([7]) Let $f \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. Consider H_N a sequence of positive linear operators from $C_B(\mathbb{R})$ into itself, $N \in \mathbb{N}$, such that $H_N(1)=1$.*

Assume that $H_N(|\cdot-x|)(x)>0$. Then

$$\left|H_N(f)(x)-f(x)\right| \leq 2\omega_1\left(f, H_N(|\cdot-x|)(x)\right), \quad \forall\ N \in \mathbb{N}. \tag{74}$$

Given that $\lim_{N\to+\infty} H_N(|\cdot-x|)(x)=0$, and f is also uniformly continuous, we obtain $\lim_{N\to+\infty} H_N(f)(x)=f(x)$.

An application of Theorem 4.3 comes next.

Theorem 4.4. *Let $f \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. Here $\theta_N=B_N$, C_N, D_N. Then*

$$\left|\theta_N(f)(x)-f(x)\right| \leq 2\omega_1\left(f, \theta_N(|\cdot-x|)(x)\right)<+\infty, \tag{75}$$

$\forall\ N \in \mathbb{N} : N^{1-\beta}>2$, $0<\beta<1$.

If f is also uniformly continuous we derive $\lim_{N\to+\infty} \theta_N(f)(x)=f(x)$.

Proof. Direct application of Theorem 4.3 and Lemmas 3.1-3.3.

Corollary 4.5. Let $f \in C_B(\mathbb{R})$. Then

$$\left\|B_N(f)-f\right\|_\infty \leq 2\omega_1\left(f,\left[\frac{1}{N^\beta}+\frac{1}{N}\left(\frac{4\left(q+\frac{1}{q}\right)}{\lambda \ln A}\right) A^{-\frac{\lambda}{2}\left(N^{1-\beta}-3\right)}\right]\right)<+\infty, \tag{76}$$

$\forall\ N \in \mathbb{N} : N^{1-\beta}>2$, $0<\beta<1$; $q,\lambda>0$, $A>1$.

If f is also uniformly continuous, we obtain $\lim_{N\to+\infty} B_N(f)=f$, pointwise and uniformly.

Proof. By Lemma 3.1 and Theorem 4.4.

We finish with the following result.

Corollary 4.6. Let $f \in C_B(\mathbb{R})$. Then

$$\left\{ \begin{array}{c} \|C_N(f) - f\|_\infty \\ \|B_N(f) - f\|_\infty \end{array} \right\} \leq$$

(77)

$$2\omega_1 \left(f, \left[\left(\frac{1}{N^\beta} + \frac{1}{N} \right) + \frac{1}{N} \left\{ \gamma A^{-\lambda\left(N^{1-\beta}-2\right)} + \left(\frac{4\left(q + \frac{1}{q}\right)}{\lambda \ln A} \right) A^{-\frac{\lambda}{2}\left(N^{1-\beta}-3\right)} \right\} \right] \right) < +\infty,$$

$\forall\ N \in \mathbb{N} : N^{1-\beta} > 2$, $0 < \beta < 1$; $q, \lambda > 0$, $A > 1$, $\gamma := 2\max\left\{q, \frac{1}{q}\right\}$.

If f is also uniformly continuous, we derive that

$$\lim_{N\to+\infty} C_N(f) = \lim_{N\to+\infty} D_N(f) = f,$$

pointwise and uniformly.

Proof. By Lemmas 3.2, 3.3 and Theorem 4.4.

REFERENCES

[1] G.A. Anastassiou, *A "K-Attainable" inequality related to the convergence of Positive Linear Operators*, J. Approximation Theory, 44, 380-383 (1985).

[2] G.A. Anastassiou, *Moments in probability and approximation theory*, Pitman Research Notes in Mathematics series / Longman Scientific & Technical, Essex, New York, UK, USA, 1993.

[3] G.A. Anastassiou, *Quantitative Approximations*, Chapmen & Hall/CRC, London, New York, 2001.

[4] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[5] G.A. Anastassiou, *Parametrized, Deformed and General Neural Networks*, Springer, Heidelberg, New York, 2023.

[6] G.A. Anastassiou, *Trigonometric and Hyperbolic Generated Approximation Theor*y, World Scientific, Singapore, New York, 2025.

[7] R.A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators*, Lecture Notes in Mathematics, Vol. 293, Springer, Heidelberg, New York, 1972.

[8] O. Shisha and B. Mond, *The degree of convergence of sequences of linear positive operators*, Nat. Acad. Sci. U.S. 60 (1968), 1196-1200.

[9] O. Shisha and B. Mond, *The degree of Approximation to Periodic Functions by Linear Positive Operator*s, J. Approximation Theory, 1, 335-339 (1968).

[10] Z. Chen and F. Cao, *The approximation operators with sigmoidal functions*, Computers and Mathematics with Applications, 58 (2009), 758-765.

[11] D. Costarelli, R. Spigler, *Approximation results for neural network operators activated by sigmoidal functions*, Neural Networks 44 (2013), 101-106.

[12] D. Costarelli, R. Spigler, *Multivariate neural network operators with sigmoidal activation functions*, Neural Networks 48 (2013), 72-77.

[13] S. Haykin, *Neural Networks: A Comprehensive Foundation* (2 ed.), Prentice Hall, New York, 1998.

[14] W. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bulletin of Mathematical Biophysics, 7 (1943), 115-133.

[15] T.M. Mitchell, *Machine Learning*, WCB-McGraw-Hill, New York, 1997.

[16] Yu, Dansheng; Cao, Feilong, *Construction and approximation rate for feed-forward neural network operators with sigmoidal functions*, J. Comput. Appl. Math. 453 (2025), paper No. 116150.

[17] Cen, Siyu; Jin, Bangti; Quan, Qimeng; Zhou, Zhi; *Hybrid neural-network FEM approximation of diffusion coeficient in elyptic and parabolic problems.* IMA J. Numer. Anal. 44 (2024), no. 5, 3059-3093.

[18] Coroianu Lucian; Costarelli, Danillo; Natale, Mariarosaria; Pantiş, Alexandra; *The approximation capabilities of Durrmeyer-type neural network operators.* J. Appl. Math. Comput. 70 (2024), no. 5, 4581-4599.

[19] Warin, Xavier; *The GroupMax neural network approximation of convex functions.* IEEE Trans. Neural Netw. Learn. Syst. 35 (2024), no. 8, 11608-11612.

[20] Fabra, Arnau; Guasch, Oriol; Baiges, Joan; Codina, Ramon; *Approximation of acoustic black holes with finite element mixed formulations and artificial neural network correction terms.* Finite Elem. Anal. Des. 241 (2024), paper No. 104236.

[21] Grohs, Philipp; Voigtlaender, Felix. *Proof of the theory-to-practice gap in deep learning via sampling complexity bounds for neural network approximation spaces.* Found. Comput. Math. 24 (2024), no. 4, 1085-1143.

[22] Basteri, Andrea; Trevisan, Dario; *Quantitative Gaussian approximation of randomly initialized deep neural networks.* Mach. Learn. 113 (2024), no. 9, 6373-6393.

[23] De Ryck, T.; Mishra, S. *Error analysis for deep neural network approximations of parametric hyperbolic conservation laws.* Math. Comp. 93 (2024), no. 350, 2643-2677.

[24] Liu, Jie; Zhang, Baoji; Lai, Yuyang; Fang, Liqiau. *Hull form optimization reserach based on multi-precision back-propagation neural network approximation model.* Internal. J. Numer. Methods Fluid 96 (2024), no. 8, 1445-1460.

[25] Yoo, Jihahm; Kim, Jaywon; Gim, Minjung; Lee, Haesung. *Error estimates of physics-informed neural networks for initial value problems.* J. Korean Soc. Ind. Appl. Math. 28 (2024), no. 1, 33-58.