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FOURTH ORDER STRONGLY NONCANONICAL NEUTRAL DIFFERENCE EQUATIONS: NEW OSCILLATION RESULTS

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ABSTRACT. The authors study the oscillatory behavior of the fourth-order neutral delay difference equation

$$\Delta(m_3(\ell)\Delta(m_2(\ell)\Delta(m_1(\ell)\Delta z(\ell)))) + q(\ell)f(y(\ell-\tau)) = 0$$

where $z(\ell) = y(\ell) + p(\ell)y(\ell-\sigma)$, under the conditions that $\sum_{\ell=\ell_0}^{\infty} m_j^{-1}(s) < \infty$, $j = 1, 2, 3$. New oscillation criteria are obtained with relatively few conditions. The results established are new to the literature as is shown through some examples.

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1. **Introduction**

This paper concerns the noncanonical fourth-order neutral delay difference equation of the form

$$\text{(E)} \qquad D_4 z(\ell) + q(\ell)f(y(\ell-\tau)) = 0, \ \ell \geq \ell_0 \geq 0,$$

where $D_0 z = z$, $D_j z = m_j \Delta(D_{j-1} z)$, $j = 1, 2, 3$, $D_4 z = \Delta(D_3 z)$, and $z(\ell) = y(\ell) + p(\ell)y(\ell-\sigma)$. Throughout the paper, we assume that

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(H_1) $\{m_j(\ell)\}$, $j = 1, 2, 3$ are positive real sequences for all integers $\ell \geq \ell_0$ and satisfy

$$M_j(\ell_0) = \sum_{\ell=\ell_0}^{\infty} \frac{1}{m_j(\ell)} < \infty, \ j = 1, 2, 3;$$

(H_2) $\{p(\ell)\}$ and $\{q(\ell)\}$ are positive real sequences with $0 \leq p(\ell) \leq p < 1$ for all $\ell \geq \ell_0$;

(H_3) σ and τ are positive integers;

(H_4) $f \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing and $\frac{f(x)}{x} \geq L > 0$ for all $x \neq 0$.

Let $\theta = \max\{\sigma, \tau\}$. By a *solution* of (E), we mean a real sequence $\{y(\ell)\}$ defined for all $\ell \geq \ell_0 - \theta$ and which satisfies (E) for all $\ell \geq \ell_0$. We consider only such solution that are nontrivial for all large ℓ. A solution of (E) is called *oscillatory* if it is neither eventually positive nor eventually negative; otherwise, it is called *nonoscillatory*. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Fourth order difference equations arise naturally in discrete-type models relating to physical, biological, and chemical phenomena, such as, problems in elasticity, deformation of structures, or soil settlement (see, for example, [1, 9]). Furthermore, in engineering and population dynamics problems, the existence of oscillatory solutions play an important role. During the past several years, there has been an increasing interest in obtaining conditions for the oscillation of solutions of different classes of fourth-order difference equations with or without deviating arguments; see [2–8, 10–21, 23–25] and the references cited therein. In particular, the authors in [4, 10, 11, 15, 19, 20, 23, 25] studied equation (E) in case

$$M_1(\ell_0) = M_2(\ell_0) = M_3(\ell_0) = \infty, \tag{1.1}$$

or

$$M_1(\ell_0) = M_2(\ell_0) = \infty, \quad \text{and} \quad M_3(\ell_0) < \infty, \tag{1.2}$$

or

$$M_1(\ell_0) = \infty, \quad M_2(\ell_0) < \infty, \quad \text{and} \quad M_3(\ell_0) = \infty \tag{1.3}$$

holds. Following the terminology introduced in [10], we say that equation (E) is in canonical form if (1.1) holds and is in semi-canonical form if (1.2), or (1.3), or

$$M_1(\ell_0) < \infty, \quad \text{and} \quad M_2(\ell_0) = M_3(\ell_0) = \infty \tag{1.4}$$

holds.

In [5], the authors studied equation (E) with $p(\ell) \equiv 0$ and established some oscillation criteria under the condition (H_1). From a review of the literature, it seems that there is nothing known about the oscillation of (E) if condition (H_1) holds, and this is due to the fact that finding relationships between a solution $\{y(\ell)\}$ and the corresponding sequence $\{z(\ell)\}$ seems to be very difficult. Motivated by this observation, in this paper our aim is to fill this gap by presenting some new criteria for the oscillation of all solutions of (E). Examples are provided to show the importance of our main results.

2. **Main Results**

For the sake of brevity, we introduce the following notation. For $\ell \geq \ell_0$, let

$$M_{12}(\ell) = \sum_{s=\ell}^{\infty} \frac{M_2(s)}{m_1(s)}, \quad M_{23}(\ell) = \sum_{s=\ell}^{\infty} \frac{M_3(s)}{m_2(s)},$$

and

$$M_{123}(\ell) = \sum_{s=\ell}^{\infty} \frac{M_{23}(s)}{m_1(s)}.$$

In what follows, we only need to consider eventually positive solutions of (E), since if $\{y(\ell)\}$ satisfies (E), then so does $\{-y(\ell)\}$.

We begin with the following classification type lemma.

Lemma 2.1. *Let $\{y(\ell)\}$ be an eventually positive solution of (E). Then there is an integer $\ell_1 \geq \ell_0$ such that the corresponding sequence $\{z(\ell)\}$ is positive and satisfies one of the following cases:*

(1) $D_1z(\ell) > 0, \;\; D_2z(\ell) > 0, \;\; D_3z(\ell) > 0, \;\; D_4z(\ell) \leq 0,$

(2) $D_1z(\ell) > 0, \;\; D_2z(\ell) > 0, \;\; D_3z(\ell) < 0, \;\; D_4z(\ell) \leq 0,$

(3) $D_1z(\ell) > 0, \;\; D_2z(\ell) < 0, \;\; D_3z(\ell) > 0, \;\; D_4z(\ell) \leq 0,$

(4) $D_1z(\ell) > 0, \;\; D_2z(\ell) < 0, \;\; D_3z(\ell) < 0, \;\; D_4z(\ell) \leq 0,$

(5) $D_1z(\ell) < 0, \;\; D_2z(\ell) > 0, \;\; D_3z(\ell) > 0, \;\; D_4z(\ell) \leq 0,$

(6) $D_1z(\ell) < 0, \;\; D_2z(\ell) > 0, \;\; D_3z(\ell) < 0, \;\; D_4z(\ell) \leq 0,$

(7) $D_1z(\ell) < 0, \;\; D_2z(\ell) < 0, \;\; D_3z(\ell) > 0, \;\; D_4z(\ell) \leq 0,$

(8) $D_1z(\ell) < 0, \;\; D_2z(\ell) < 0, \;\; D_3z(\ell) < 0, \;\; D_4z(\ell) \leq 0,$

for all $\ell \geq \ell_1$.

Proof. The proof is obvious and so the details are omitted. (The reader might also wish to consult [1] among other references.) □

In the following lemmas, we find relationships between the sequence $\{y(\ell)\}$ and the corresponding sequence $\{z(\ell)\}$ if the cases (1)–(8) in Lemma 2.1 are satisfied. These are essential for obtaining our oscillation criteria for (E).

Lemma 2.2. *Let $\{y(\ell)\}$ be an eventually positive solution of* (E) *with the corresponding sequence $\{z(\ell)\}$ satisfying cases (1)–(4) of Lemma 2.1. Then*

$$y(\ell) \geq (1 - p(\ell))z(\ell) \tag{2.1}$$

for all $\ell \geq \ell_1 \geq \ell_0$.

Proof. From the cases (1)–(4), we see that $z(\ell)$ is positive and increasing, and by the definition of $z(\ell)$, we have

$$\begin{aligned} y(\ell) &= z(\ell) - p(\ell)y(\sigma(\ell)) \geq z(\ell) - p(\ell)z(\sigma(\ell)) \\ &\geq (1 - p(\ell))z(\ell), \ \ell \geq \ell_1 \geq \ell_0. \end{aligned}$$

This proves the lemma. □

Lemma 2.3. *Let $\{y(\ell)\}$ be an eventually positive solution of* (E) *with the sequence $\{z(\ell)\}$ satisfying case (5) of Lemma 2.1. Then*

$$y(\ell) \geq \left(1 - \frac{p(\ell)M_{12}(\ell - \sigma)}{M_{12}(\ell)}\right) z(\ell) \tag{2.2}$$

for all $\ell \geq \ell_1 \geq \ell_0$.

Proof. In view of case (5) of Lemma 2.1, we see that

$$-D_1 z(\ell) \geq D_1 z(\infty) - D_1 z(\ell) = \sum_{s=\ell}^{\infty} \frac{1}{m_2(s)} D_2 z(s) \geq M_2(\ell) D_2 z(\ell), \tag{2.3}$$

so

$$\Delta\left(\frac{-D_1 z(\ell)}{M_2(\ell)}\right) = -\frac{(M_2(\ell)D_2 z(\ell) + D_1 z(\ell))}{m_2(\ell)M_2(\ell)M_2(\ell+1)} \geq 0.$$

Therefore,

$$\frac{-D_1 z(\ell)}{M_2(\ell)} \quad \text{is nondecreasing.} \tag{2.4}$$

Now, using (2.4), we have

$$z(\ell) \geq -\sum_{s=\ell}^{\infty} \frac{M_2(s) D_1 z(s)}{m_1(s) M_2(s)} \geq \frac{-D_1 z(\ell)}{M_2(\ell)} M_{12}(\ell)$$

and

$$\Delta\left(\frac{z(\ell)}{M_{12}(\ell)}\right) = \frac{M_{12}(\ell)D_1 z(\ell) + M_2(\ell) z(\ell)}{m_1(\ell)M_{12}(\ell)M_{12}(\ell+1)} \geq 0.$$

Hence,

$$\frac{z(\ell)}{M_{12}(\ell)} \quad \text{is nondecreasing.} \tag{2.5}$$

From the definition of $z(\ell)$ and (2.5), we have

$$y(\ell) \geq z(\ell) - p(\ell)z(\ell - \sigma) \geq \left(1 - \frac{p(\ell)M_{12}(\ell - \sigma)}{M_{12}(\ell)}\right) z(\ell), \quad \ell \geq \ell_1,$$

and this completes the proof. □

Lemma 2.4. *Let $\{y(\ell)\}$ be an eventually positive solution of* (E) *with the sequence $\{z(\ell)\}$ satisfying case (6) of Lemma 2.1. Then*

$$y(\ell) \geq \left(1 - \frac{p(\ell)M_{123}(\ell-\sigma)}{M_{123}(\ell)}\right) z(\ell) \tag{2.6}$$

for all $\ell \geq \ell_1 \geq \ell_0$.

Proof. In view of case (6) of Lemma 2.1, we see that

$$D_2 z(\ell) - D_2 z(\infty) = -\sum_{s=\ell}^{\infty} \frac{1}{m_3(s)} D_3 z(s) \geq -M_3(\ell) D_3 z(\ell). \tag{2.7}$$

Hence,

$$\Delta\left(\frac{D_2 z(\ell)}{M_3(\ell)}\right) = \frac{M_3(\ell)D_3 z(\ell) + D_2 z(\ell)}{m_3(\ell)M_3(\ell)M_3(\ell+1)} \geq 0,$$

which shows that $\left\{\frac{D_2 z(\ell)}{M_3(\ell)}\right\}$ is nondecreasing. Using this property, we see that

$$-D_1 z(\ell) \geq \sum_{s=\ell}^{\infty} \frac{1}{m_2(s)} D_2 z(s) \geq \frac{D_2 z(s)}{M_3(\ell)} \sum_{s=\ell}^{\infty} \frac{M_3(s)}{m_2(s)} = \frac{M_{23}(\ell)}{M_3(\ell)} D_2 z(\ell). \tag{2.8}$$

By (2.8), we have

$$\Delta\left(\frac{-D_1 z(\ell)}{M_{23}(\ell)}\right) = \frac{-M_{23}(\ell)D_2 z(\ell) - M_3(\ell)D_1 z(\ell)}{m_2(\ell)M_{23}(\ell)M_{23}(\ell+1)} \geq 0,$$

and so $\left\{\frac{-D_1 z(\ell)}{M_{23}(\ell)}\right\}$ is nondecreasing. We also have

$$z(\ell) \geq -\sum_{s=\ell}^{\infty} \frac{1}{m_1(s)} D_1 z(s) \geq \frac{-D_1 z(\ell)}{M_{23}(\ell)} \sum_{s=\ell}^{\infty} \frac{M_{23}(s)}{m_1(s)} = \frac{-M_{123}(\ell)}{M_{23}(\ell)} D_1 z(\ell). \tag{2.9}$$

Hence, by (2.9), we obtain that

$$\Delta\left(\frac{z(\ell)}{M_{123}(\ell)}\right) = \frac{M_{123}(\ell)D_1 z(\ell) + M_{23}(\ell)z(\ell)}{m_1(\ell)M_{123}(\ell)M_{123}(\ell+1)} \geq 0,$$

and so

$$\frac{z(\ell)}{M_{123}(\ell)} \quad \text{is nondecreasing.} \tag{2.10}$$

From the definition of $z(\ell)$ and (2.10), we see that

$$y(\ell) \geq z(\ell) - p(\ell)z(\ell-\sigma) \geq \left(1 - \frac{p(\ell)M_{123}(\ell-\sigma)}{M_{123}(\ell)}\right) z(\ell), \quad \ell \geq \ell_1,$$

which completes the proof. □

Lemma 2.5. *Let $\{y(\ell)\}$ be an eventually positive solution of* (E) *with the sequence $\{z(\ell)\}$ satisfying case (7) or case (8) of Lemma 2.1. Then*

$$y(\ell) \geq \left(1 - \frac{p(\ell)M_1(\ell-\sigma)}{M_1(\ell)}\right) z(\ell) \tag{2.11}$$

for all $\ell \geq \ell_1 \geq \ell_0$.

Proof. Assume that case (7) or case (8) of Lemma 2.1 holds. In both of these cases we see that $z(\ell) > 0$ and $D_1 z(\ell)$ is decreasing for all $\ell \geq \ell_1$. Using this, we see that

$$z(\ell) \geq z(\ell) - z(\infty) = -\sum_{s=\ell}^{\infty} \frac{D_1 z(s)}{m_1(s)} \geq -M_1(\ell) D_1 z(\ell).$$

Hence,

$$\Delta\left(\frac{z(\ell)}{M_1(\ell)}\right) = \frac{M_1(\ell) D_1 z(\ell) + z(\ell)}{m_1(\ell) M_1(\ell) M_1(\ell+1)} \geq 0,$$

which shows that $\left\{\frac{z(\ell)}{M_1(\ell)}\right\}$ is nondecreasing. Using this property in the definition of $z(\ell)$, we have

$$y(\ell) \geq z(\ell) - p(\ell) z(\ell - \sigma) \geq \left(1 - \frac{p(\ell) M_1(\ell - \sigma)}{M_1(\ell)}\right) z(\ell),$$

for all $\ell \geq \ell_1$. This completes the proof of the lemma. □

Remark 2.6. Let us define

$$d(\ell) = \min\left\{p(\ell), \frac{p(\ell) M_1(\ell-\sigma)}{M_1(\ell)}, \frac{p(\ell) M_{12}(\ell-\sigma)}{M_{12}(\ell)}, \frac{p(\ell) M_{123}(\ell-\sigma)}{M_{123}(\ell)}\right\};$$

then from (2.1), (2.2), (2.6) and (2.11), we have that the relation

$$y(\ell) \geq (1 - d(\ell)) z(\ell)$$

holds. We further assume going forward that $(1 - d(\ell)) > 0$ for all $\ell \geq \ell_1 \geq \ell_0$.

Lemma 2.7. *Let $\{y(\ell)\}$ be an eventually positive solution of* (E) *with the corresponding sequence $\{z(\ell)\}$ satisfying any of the cases (1)–(8) of Lemma 2.1. Then*

$$D_4 z(\ell) + L q(\ell)(1 - d(\ell - \tau)) z(\ell - \tau) \leq 0 \tag{E_1}$$

for all $\ell \geq \ell_1 \geq \ell_0$.

Proof. Let $\{y(\ell)\}$ be an eventually positive solution of (E); then there exists an integer $\ell_1 \geq \ell_0$ such that $y(\ell - \sigma) > 0$ and $y(\ell - \tau) > 0$ for all $\ell \geq \ell_1$. From Lemmas 2.2 to 2.5, combined with Remark 2.6, we see that

$$y(\ell - \tau) \geq (1 - d(\ell - \tau)) z(\ell - \tau), \quad \ell \geq \ell_1,$$

and using this in (E) along with condition (H_4), we obtain (E_1). This proves the lemma. □

Next, we define

$$Q(\ell, \ell_*) = \sum_{s=\ell_*}^{\ell-1} \frac{1}{m_2(s)} \sum_{t=\ell_*}^{s-1} \frac{1}{m_3(t)} \sum_{j=\ell_*}^{t-1} q(j)(1 - d(j - \tau))$$

and

$$\overline{Q}(\ell, \ell_*) = \sum_{s=\ell_*}^{\ell-1} \frac{q(s) M_{123}(s - \tau)}{M_3(s - \tau)} (1 - d(s - \tau))$$

for all $\ell \geq \ell_*$ and any $\ell_* \geq \ell_0$.

Lemma 2.8. *Let $\{y(\ell)\}$ be an eventually positive solution of $\ell \geq \ell_*$. If*

$$Q(\infty, \ell_0) = \infty, \tag{2.12}$$

then the sequence $\{z(\ell)\}$ cannot satisfy any of the cases (1)–(4) in Lemma 2.1.

Proof. From (H_1) and (2.12), we can see that

$$\sum_{\ell=\ell_0}^{\infty} \frac{1}{m_3(\ell)} \sum_{s=\ell_0}^{\ell-1} q(s)(1-d(s-\tau)) = \sum_{\ell=\ell_0}^{\infty} q(\ell)(1-d(\ell-\tau)) = \infty. \tag{2.13}$$

Now assume that $\{y(\ell)\}$ is an eventually positive solution of (E). Then there exists an integer $\ell_1 \geq \ell_0$ such that $y(\ell-\sigma) > 0$ and $y(\ell-\tau) > 0$ for all $\ell \geq \ell_1$. From the definition of $z(\ell)$, we see that $z(\ell) > 0$ and satisfies cases (1)–(4) of Lemma 2.1. In view of Lemma 2.7, we see that

$$D_4 z(\ell) + Lq(\ell)(1-d(\ell-\tau))z(\ell-\tau) \leq 0, \quad \ell \geq \ell_1. \tag{2.14}$$

Since in all four cases, $\{z(\ell)\}$ is increasing, there is a constant $c > 0$ and an integer $\ell_2 \geq \ell_1$ such that $z(\ell-\tau) \geq c$ for all $\ell \geq \ell_2$. Using this in (2.14), we obtain

$$-D_4 z(\ell) \geq Lcq(\ell)(1-d(\ell-\tau)), \ \ell \geq \ell_2. \tag{2.15}$$

Summing (2.15) from ℓ_2 to $\ell - 1$ gives

$$-D_3 z(\ell) + D_3 z(\ell_2) \geq Lc \sum_{s=\ell_2}^{\ell-1} q(s)(1-d(s-\tau)). \tag{2.16}$$

If we assume that $\{z(\ell)\}$ satisfies either case (1) or case (3), then from (2.13) and (2.16),

$$D_3 z(\ell_2) \geq Lc \sum_{s=\ell_2}^{\ell-1} q(s)(1-d(s-\tau)) \to \infty \ \text{ as } \ \ell \to \infty, \tag{2.17}$$

which is a contradiction.

Next, assume that case (2) holds. From (2.16), we see that

$$-D_3 z(\ell) \geq Lc \sum_{s=\ell_2}^{\ell-1} q(s)(1-d(s-\tau))$$

or

$$-\Delta(D_2 z(\ell)) \geq \frac{Lc}{m_3(\ell)} \sum_{s=\ell_2}^{\ell-1} q(s)(1-d(s-\tau)). \tag{2.18}$$

Summing (2.18) from ℓ_2 to $\ell - 1$, we obtain

$$D_2 z(\ell_2) - D_2 z(\ell) \geq Lc \sum_{s=\ell_2}^{\ell-1} \frac{1}{m_3(s)} \sum_{t=\ell_2}^{s-1} q(t)(1-d(t-\tau)),$$

which, in view of (2.13), gives

$$D_2z(\ell_2) \geq Lc\sum_{s=\ell_2}^{\ell-1}\frac{1}{m_3(s)}\sum_{t=\ell_2}^{s-1}q(t)(1-d(t-\tau)) \to \infty \text{ as } \ell \to \infty. \tag{2.19}$$

This is clearly a contradiction.

Finally, assume that case (4) holds. Proceeding as in the last case, we have from (2.19) that

$$-\Delta(D_1z(\ell)) \geq \frac{Lc}{m_2(\ell)}\sum_{s=\ell_2}^{\ell-1}\frac{1}{m_3(s)}\sum_{t=\ell_2}^{s-1}q(t)(1-d(t-\tau)).$$

Summing from ℓ_2 to $\ell-1$,

$$D_1z(\ell_2)-D_1z(\ell) \geq Lc\sum_{s=\ell_2}^{\ell-1}\frac{1}{m_2(s)}\sum_{t=\ell_2}^{s-1}\frac{1}{m_3(t)}\sum_{j=\ell_2}^{t-1}q(j)(1-d(j-\tau)) = LcQ(\ell,\ell_2). \tag{2.20}$$

In view of (2.12), this implies that $D_1z(\ell_2) \geq LcQ(\ell,\ell_2) \to \infty$ as $\ell \to \infty$, and this contradiction completes the proof. □

Next, in our first main result, we show that under a simple condition, any nonoscillatory solution of (E) converges to zero as $\ell \to \infty$.

Theorem 2.9. *Assume that* (H_1)*–*(H_4) *hold. If*

$$\sum_{\ell=\ell_0}^{\infty}\frac{Q(\ell,\ell_0)}{m_1(\ell)} = \infty, \tag{2.21}$$

then any solution $\{y(\ell)\}$ *of* (E) *is either oscillatory or* $\lim_{\ell\to\infty} y(\ell) = 0$.

Proof. Let $\{y(\ell)\}$ be an eventually positive solution of (E). Then there exists an integer $\ell_1 \geq \ell_0$ such that $y(\ell-\sigma) > 0$ and $y(\ell-\tau) > 0$ for all $\ell \geq \ell_1$. Then, $z(\ell) > 0$, and by Lemma 2.1, eight possible cases may occur for $\ell \geq \ell_1$.

Since (2.20) together with (H_1) imply that $\sum_{\ell=\ell_0}^{\infty} Q(\ell,\ell_0)$ cannot be bounded, by Lemma 2.8, cases (1)–(4) cannot hold.

Assume one of the cases (5)–(8) holds. Since z is decreasing, $z(\infty) = \lim_{\ell\to\infty} z(\ell) = c_0$ with $0 \leq c_0 < \infty$. Assume that $c_0 > 0$. Then there is an integer $\ell_2 \geq \ell_1$ such that $z(\ell-\tau) \geq c_0$ for $\ell \geq \ell_2$, and from (E_1), we have

$$-D_4z(\ell) \geq Lc_0q(\ell)(1-d(\ell-\tau)).$$

Then we can easily arrive at the contradiction (2.17) in cases (5) and (7), and the contradiction (2.19) in case (6). Hence, we conclude that $c_0 = 0$.

If we assume that case (8) holds, then we arrive at (2.20), that is,

$$-D_1z(\ell) \geq Lc_0Q(\ell,\ell_2),$$

or

$$-\Delta z(\ell) \geq \frac{Lc_0}{m_1(\ell)} Q(\ell, \ell_2).$$

Summing the last inequality from ℓ_2 to $\ell - 1$ gives

$$z(\ell_2) \geq Lc \sum_{s=\ell_2}^{\ell-1} \frac{Q(s, \ell_2)}{m_1(s)}.$$

But in view of (2.21), the summation on the right-hand side of the last inequality tends to ∞ as $\ell \to \infty$, which is a contradiction. Hence, $\lim_{\ell\to\infty} z(t) = 0$, and since $y(\ell) \leq z(\ell)$, this implies that $\lim_{\ell\to\infty} y(\ell) = 0$. This completes the proof of the theorem. □

In the sequel, we present two theorems on for the oscillation of all solutions of (E).

Theorem 2.10. *Assume that* (H_1)*–*(H_4) *hold. If*

$$\limsup_{\ell\to\infty} Q_1(\ell, \ell_1) > \frac{1}{L} \tag{2.22}$$

for any integer $\ell_1 \geq \ell_0$*, where*

$$Q_1(\ell, \ell_1) = \min\{M_1(\ell)Q(\ell, \ell_1), M_3(\ell)\overline{Q}(\ell, \ell_1)\},$$

then (E) *is oscillatory.*

Proof. Let $\{y(\ell)\}$ be an eventually positive solution of (E); then there exists an integer $\ell_1 \geq \ell_0$ such that $y(\ell - \sigma) > 0$ and $y(\ell - \tau) > 0$ for all $\ell \geq \ell_1$. Now, the sequence $z(\ell) > 0$, and by Lemma 2.1, eight possible cases may occur for $\ell \geq \ell_1$.

First note that, in view of (H_1), in order for (2.22) to hold, we must have that

$$Q(\infty, \ell_0) = \overline{Q}(\infty, \ell_0) = \infty. \tag{2.23}$$

In view of Lemma 2.8, we see that condition (2.23) ensures that cases (1)–(4) of Lemma 2.1 are impossible. Hence, we shall consider the remaining possible cases (5)–(8) individually.

Assume that case (5) holds. From Lemma 2.7,

$$D_4 z(\ell) + Lq(\ell)(1 - d(\ell - \tau))z(\ell - \tau) \leq 0, \tag{2.24}$$

and from (2.3),

$$-D_1 z(\ell) \geq M_2(\ell) D_2 z(\ell)$$

or

$$-\Delta z(\ell) \geq \frac{M_2(\ell)}{m_1(\ell)} D_2 z(\ell).$$

Summing the above inequality from ℓ to ∞ gives

$$z(\ell) \geq D_2 z(\ell) \sum_{s=\ell}^{\infty} \frac{M_2(s)}{m_1(s)} = M_{12}(\ell) D_2 z(\ell). \tag{2.25}$$

Using (2.25) and the increasing property of $D_2 z(\ell)$ in (2.24), there exists a constant $c_1 > 0$ and an integer $\ell_2 \geq \ell_1$ such that

$$-D_4 z(\ell) \geq L c_1 q(\ell)(1 - d(\ell - \tau)) M_{12}(\ell - \tau), \quad \ell \geq \ell_2.$$

Summing from ℓ_2 to $\ell - 1$, we obtain

$$D_3 z(\ell_2) \geq D_3 z(\ell) + L c_1 \sum_{s=\ell_2}^{\ell-1} q(s)(1 - d(s - \tau)) M_{12}(s - \tau). \tag{2.26}$$

Taking (H_1) and (2.23) into account, it is easy to see that

$$\begin{aligned} \infty = \overline{Q}(\infty, \ell_0) &= \sum_{\ell=\ell_0}^{\infty} q(s)(1 - d(\ell - \tau)) \frac{M_{123}(\ell - \tau)}{M_3(\ell - \tau)} \\ &\leq \sum_{\ell=\ell_0}^{\infty} q(\ell)(1 - d(\ell - \tau)) M_{12}(\ell - \tau). \end{aligned} \tag{2.27}$$

Using (2.27) in (2.26), we obtain a contradiction.

Assume that case (6) holds. From Lemma 2.7, we get (2.24). From (2.8) and (2.9), we have

$$z(\ell) \geq \frac{M_{123}(\ell)}{M_3(\ell)} D_2 z(\ell), \tag{2.28}$$

and using (2.28) in (2.24) gives

$$-D_4 z(\ell) \geq L q(\ell)(1 - d(\ell - \tau)) \frac{M_{123}(\ell - \tau)}{M_3(\ell - \tau)} D_2 z(\ell - \tau). \tag{2.29}$$

Summing (2.29) from ℓ_1 to $\ell - 1$ and using the monotonicity of $D_2 z(\ell)$, we find that

$$-D_3 z(\ell) \geq L D_2(\ell - \tau) \sum_{s=\ell_1}^{\ell-1} q(s)(1 - d(s - \tau)) \frac{M_{123}(s - \tau)}{M_1(s - \tau)} \geq L D_2 z(\ell) \overline{Q}(\ell, \ell_1). \tag{2.30}$$

From (2.7) and (2.30), we obtain

$$-D_3 z(\ell) \geq -L M_3(\ell) \overline{Q}(\ell, \ell_1) D_3 z(\ell).$$

Dividing both sides by $-D_3 z(\ell)$ and then taking the lim sup as $\ell \to \infty$ on both sides of the resulting inequality, contradicts (2.22).

Now assume that case (7) holds. From Lemma 2.7, we have (2.28). Summing (2.24) from ℓ_1 to $\ell - 1$ and using the fact that $\{\frac{z(\ell)}{M_1(\ell)}\}$ is nonincreasing, we have

$$\begin{aligned} D_3 z(\ell_1) &\geq D_3 z(\ell) + L \sum_{s=\ell_1}^{\ell-1} q(s)(1 - d(s - \tau)) z(s - \tau) \\ &\geq \frac{z(\ell_1)}{M_1(\ell_1)} L \sum_{s=\ell_1}^{\ell-1} q(s)(1 - d(s - \tau)) M_1(s - \tau). \end{aligned} \tag{2.31}$$

On the other hand, using (H_1) and (2.27), it is easy to see that for any constant $c_2 > 0$,

$$\infty = \sum_{\ell=\ell_1}^{\infty} q(\ell)(1 - d(\ell - \tau))M_{12}(\ell - \tau) \le c_2 \sum_{\ell=\ell_1}^{\infty} q(\ell)(1 - d(\ell - \tau))M_1(\ell - \tau),$$

which, in view of (2.31), is again a contradiction.

Finally assume that (8) holds. Again from Lemma 2.7, we have (2.24). Summing (2.24) from ℓ_1 to $\ell - 1$,

$$-D_3 z(\ell) \ge L \sum_{s=\ell_1}^{\ell-1} q(s)(1 - d(s - \tau))z(s - \tau) \ge Lz(\ell - \tau) \sum_{s=\ell_1}^{\ell-1} q(s)(1 - d(s - \tau)).$$

Dividing the last inequality by $m_3(\ell)$ and then summing from ℓ_1 to $\ell - 1$, we obtain

$$-D_2 z(\ell) \ge Lz(\ell - \tau) \sum_{s=\ell_1}^{\ell-1} \frac{1}{m_3(s)} \sum_{t=\ell_1}^{s-1} q(t)(1 - d(t - \tau)). \tag{2.32}$$

Similarly, we can obtain

$$\begin{aligned} -D_1 z(\ell) &\ge Lz(\ell - \tau) \sum_{s=\ell_1}^{\ell-1} \frac{1}{m_2(s)} \sum_{t=\ell_1}^{s-1} \frac{1}{m_3(t)} \sum_{j=\ell_1}^{t-1} q(j)(1 - d(j - \tau)) \\ &\ge Lz(\ell)Q(\ell, \ell_1) \ge -LM_1(\ell)Q(\ell, \ell_1)D_1 z(\ell), \end{aligned}$$

that is,

$$\frac{1}{L} \ge M_1(\ell)Q(\ell, \ell_1),$$

which clearly contradicts (2.22). The proof is now complete. □

Our final theorem is obtained by using the classical Riccati transformation technique.

Theorem 2.11. *Assume that* (H_1)–(H_4) *hold. If for all sufficiently large* $\ell_1 \ge \ell_0$,

$$\limsup_{\ell\to\infty} \sum_{s=\ell_1}^{\ell} \left(Lq(s)(1 - d(s - \tau))M_{123}(s) - \frac{M_{23}(s)}{4m_1(s)M_{123}(s + 1)} \right) = \infty \tag{2.33}$$

and

(2.34)

$$\limsup_{\ell\to\infty} \sum_{s=\ell_1}^{\ell} \left(\frac{LM_1(s + 1)}{m_2(s)} \sum_{t=\ell_1}^{s-1} \frac{1}{m_3(t)} \sum_{j=\ell_1}^{t-1} q(j)(1 - d(j - \tau)) - \frac{1}{4M_1(s)m_1(s)} \right) = \infty,$$

then (E) *is oscillatory.*

Proof. Let $\{y(\ell)\}$ be an eventually positive solution of (E) such that $y(\ell - \tau) > 0$ and $y(\ell - \sigma) > 0$ for all $\ell \ge \ell_1 \ge \ell_0$. Then $z(\ell) > 0$ and by Lemma 2.1, there are eight possible cases that may occur for $\ell \ge \ell_1$.

From (2.34), we see that

$$\sum_{\ell=\ell_1}^{\infty} \frac{M_1(s)}{m_2(s)} \sum_{s=\ell_1}^{\ell-1} \frac{1}{m_3(s)} \sum_{t=\ell_1}^{s-1} q(t)(1-d(t-\tau)) = \infty,$$

which, in view of condition (H_1) implies that $Q(\infty, \ell_0) = \infty$. Thus, by Lemma 2.8, cases (1)–(4) in Lemma 2.1 cannot hold, so we will consider cases (5)–(8) one at a time.

Let case (5) hold. From (2.33), we have

$$\sum_{\ell=\ell_0}^{\infty} q(\ell)(1-d(\ell-\tau))M_{123}(\ell) = \infty.$$

Then arguing as in the proof of Theorem 2.10 for case (5), we arrive at a contradiction.

Assume that case (6) holds. Define the sequence

$$w(\ell) = \frac{D_3 z(\ell)}{z(\ell)} < 0, \quad \ell \geq \ell_1.$$

Combining (2.7) and (2.28), we obtain

$$-1 \leq w(\ell) M_{123}(\ell). \tag{2.35}$$

Also, from (2.7) and (2.8),

$$-\Delta z(\ell) \geq -\frac{M_{23}(\ell)}{m_1(\ell)} D_3 z(\ell). \tag{2.36}$$

Now by (2.24), (2.36), and the monotonicity of $\{z(\ell)\}$, we see that

$$\begin{aligned} \Delta w(\ell) &= \frac{D_4 z(\ell)}{z(\ell+1)} - \frac{D_3 z(\ell) \Delta z(\ell)}{z(\ell) z(\ell+1)} \\ &\leq -Lq(\ell)(1-d(\ell-\tau)) \frac{z(\ell-\tau)}{z(\ell+1)} - \frac{(D_3 z(\ell))^2 M_{23}(\ell)}{m_1(\ell) z^2(\ell)} \\ &\leq -Lq(\ell)(1-d(\ell-\tau)) - \frac{M_{23}(\ell)}{m_1(\ell)} w^2(\ell), \end{aligned}$$

or

$$\Delta w(\ell) + Lq(\ell)(1-d(\ell-\tau)) + \frac{M_{23}(\ell)}{m_1(\ell)} w^2(\ell) \leq 0.$$

Multiplying the above inequality by $M_{123}(\ell+1)$ and summing the resulting inequality from ℓ_1 to $\ell-1$, we obtain

$$\begin{aligned} \sum_{s=\ell_1}^{\ell-1} M_{123}(s+1)\Delta w(s) + L \sum_{s=\ell_1}^{\ell-1} M_{123}(s+1) q(s)(1-d(s-\tau)) \\ + L \sum_{s=\ell_1}^{\ell-1} M_{123}(s+1) M_{23}(s) \frac{w^2(s)}{m_1(s)} \leq 0. \end{aligned} \tag{2.37}$$

Now, applying the summation by parts formula and then rearranging terms, we have

$$w(\ell)M_{123}(\ell) - w(\ell_1)M_{123}(\ell_1) + \sum_{s=\ell_1}^{\ell-1} LM_{123}(s+1)q(s)(1-d(s-\tau))$$

$$+\sum_{s=\ell_1}^{\ell-1}\frac{M_{123}(s)}{m_1(s)}w(s) + \sum_{s=\ell_1}^{\ell-1}\frac{M_{23}(s)M_{123}(s+1)}{m_1(s)}w^2(s) \le 0. \tag{2.38}$$

Using completing the square and then applying (2.35) leads to

$$\sum_{s=\ell_1}^{\ell-1}\left(Lq(s)M_{123}(s+1)(1-d(s-\tau)) - \frac{M_{23}(s)}{4m_1(s)M_{123}(s+1)}\right)$$

$$\le w(\ell_1)M_{123}(\ell_1) + 1 < \infty,$$

which contradicts (2.33) as $\ell \to \infty$.

Assume now that case (7) holds. Note that

$$\sum_{\ell=\ell_0}^{\infty} q(\ell)(1-d(\ell-\tau))M_{123}(\ell) = \infty$$

is necessary for (2.33) to hold. Then, for any constant $c_3 > 0$, we have

$$\infty = \sum_{\ell=\ell_0}^{\infty} q(\ell)(1-d(\ell-\tau))M_{123}(\ell) \le c_3\sum_{\ell=\ell_0}^{\infty} q(\ell)(1-d(\ell-\tau))M_{12}(\ell). \tag{2.39}$$

Proceeding as in the proof of case (7) in Theorem 2.10, we obtain a contradiction.

Finally assume that case (8) holds. Define

$$v(\ell) = \frac{D_1z(\ell)}{z(\ell)} < 0.$$

From (2.32), we have

$$-D_2z(\ell) \ge Lz(\ell+1)\sum_{s=\ell_1}^{\ell-1}\frac{1}{m_3(s)}\sum_{t=\ell_1}^{s-1} q(t)(1-d(t-\tau)). \tag{2.40}$$

On the other hand, from the monotonicity of $D_1z(\ell)$, we have

$$z(\infty) - z(\ell) = \sum_{s=\ell}^{\infty}\frac{D_1z(s)}{m_1(s)} \le M_1(\ell)D_1z(\ell) \tag{2.41}$$

or

$$-1 \le v(\ell)M_1(\ell) < 0. \tag{2.42}$$

Using (2.40) we obtain

$$\Delta v(\ell) = \frac{D_2z(\ell)}{m_2(\ell)z(\ell+1)} - \frac{(D_1z(\ell))^2}{m_1(\ell)z(\ell)z(\ell+1)}$$

$$\le \frac{-L}{m_2(\ell)}\sum_{s=\ell_1}^{\ell-1}\frac{1}{m_3(s)}\sum_{t=\ell_1}^{s-1} q(t)(1-d(t-\tau)) - \frac{v^2(\ell)}{m_1(\ell)}.$$

Multiplying the last inequality by $M_1(\ell+1)$ and then summing from ℓ_1 to $\ell-1$ gives

$$v(\ell)M_1(\ell) - v(\ell_1)M_1(\ell_1) + \sum_{s=\ell_1}^{\ell-1} \frac{LM_1(s+1)}{m_2(s)} \sum_{t=\ell_1}^{s-1} \frac{1}{m_3(t)} \sum_{j=\ell_1}^{t-1} q(j)(1-d(j-\tau))$$
$$+ \sum_{s=\ell_1}^{\ell-1} \frac{v(s)}{m_1(s)} + \sum_{s=\ell_1}^{\ell-1} M_1(s+1)\frac{v^2(s)}{m_1(s)} \leq 0.$$

Completing the square and then using (2.42), we obtain

$$\sum_{s=\ell_1}^{\ell-1} \left(\frac{LM_1(s+1)}{m_2(s)} \sum_{t=\ell_1}^{s-1} \frac{1}{m_3(t)} \sum_{j=\ell_1}^{t-1} q(j)(1-d(j-\tau)) - \frac{1}{4m_1(s)M_1(s+1)} \right)$$
$$\leq 1 + M_1(\ell_1)v(\ell_1),$$

which contradicts (2.34) and completes the proof. □

3. **Examples**

In this section, we provide two examples to illustrate the applicability and novelty of our results.

Example 3.1. Consider the fourth-order neutral delay difference equation

$$\Delta(\ell(\ell+1)\Delta(\ell(\ell+1)\Delta(\ell(\ell+1)\Delta z(\ell)))) + q_0\ell^2 y(\ell-2) = 0, \ \ell \geq 2, \tag{3.1}$$

where $z(\ell) = y(\ell) + \frac{1}{16}y(\ell-1)$ and $q_0 > 0$. Here we have $m_1(\ell) = m_2(\ell) = m_3(\ell) = \ell(\ell+1)$, $p(\ell) = 1/16$, $q(\ell) = q_0\ell^2$, $\sigma = 1$, $\tau = 2$, and $f(y) = y$. Simple calculations show that $M_1(\ell) = M_2(\ell) = M_3(\ell) = 1/\ell$, $M_{12}(\ell) \approx 1/2\ell^2 = M_{23}(\ell)$, and $M_{123}(\ell) \approx 1/6\ell^3$. Furthermore, $d(\ell) = \frac{1}{16}$ and $L = 1$. It is easy to see that $Q(\ell, 2) \approx \frac{5q_0}{32}\ell \to \infty$ as $\ell \to \infty$. Therefore, by Theorem 2.9, nonoscillatory solutions of (3.1) converge to zero as $\ell \to \infty$.

Example 3.2. Consider the equation

$$\Delta(\ell(\ell+1)\Delta(\ell(\ell+1)\Delta(\ell(\ell+1)\Delta z(\ell)))) + q_0\ell^2 y(\ell-2)(1+y^2(\ell-2)) = 0, \tag{3.2}$$

where $z(\ell) = y(\ell) + \frac{1}{16}y(\ell-1)$, $q_0 > 0$, and $\ell \geq 2$. Here we have $f(y) = y(1+y^2)$, $Q(\ell, 2) = \bar{Q}(\ell, 2) \approx \frac{5q_0\ell}{32}$, and the other quantities are as in the previous example.

Condition (2.22) becomes

$$\limsup_{\ell\to\infty} \frac{5q_0\ell}{32\ell} = \frac{5q_0}{32} > 1,$$

so that (2.22) is satisfied if $q_0 > 6.4$. Therefore, by Theorem 2.10, equation (3.2) is oscillatory if $q_0 > 6.4$. The same conclusion follows from Theorem 2.11 if $q_0 > 4.8$. Thus, Theorem 2.11 gives a better condition than Theorem 2.10.

4. Conclusion

In this paper, we first found the relationship between a solution $\{y(\ell)\}$ and the sequence $\{z(\ell)\}$ in the case where $z(\ell)$ satisfies one of the eight different conditions in Lemma 2.1. Using these relationships, we obtained sufficient conditions for the oscillation of all solutions of (E). Notice that using the method developed in this paper, it would also be possible to obtain oscillation criteria for equation (E) in the cases where the equation is one of the semi-noncanonical types

$$M_1(\ell_0) = \infty, \quad M_2(\ell_0) < \infty, \quad \text{and} \quad M_3(\ell_0) < \infty,$$

or

$$M_1(\ell_0) < \infty, \quad M_2(\ell_0) < \infty, \quad \text{and} \quad M_3(\ell_0) = \infty,$$

or

$$M_1(\ell_0) < \infty, \quad M_2(\ell_0) = \infty, \quad \text{and} \quad M_3(\ell_0) < \infty.$$

This is left for future research. Also notice that none of the results currently in the literature when applied to equations (3.1) or (3.2) can yield the oscillation of all solutions since these equations are not canonical. It would be of interest to extend the results here to the cases where the neutral coefficient $p(\ell) \geq 1$, p is nonpositive, or if p is unbounded. (For a discussion of how different values of p affect the oscillatory behavior of solutions of neutral difference equations, we refer the reader to [22].)

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