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FUZZY SYMMETRIZED AND PERTURBED NEURAL NETWORK APPROXIMATION

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ABSTRACT. Here we treat the univariate fuzzy quantitative approximation of fuzzy real valued functions on a compact interval by quasi-interpolation symmetrized and perturbed hyperbolic tangent and generalized logistic functions based fuzzy neural network operators. These approximations are derived by establishing fuzzy Jackson type inequalities involving the fuzzy modulus of continuity of the involved fuzzy function and the fuzzy derivative of the engaged function. The approximations are fuzzy pointwise and fuzzy uniform. The related feed-forward fuzzy neural networks are with one hidden layer. The foundation of this work is based on positive linear operators generated deterministic real neural networks approximations.

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1. Introduction

The author in [1] and [2], see Chapters 2-5 was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types ([12]), by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions were assumed to be of compact support.

The author inspired by [13], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of various sigmoidal activation functions resulted into [5]-[9], by treating both the univariate and multivariate cases.

In Section 2, we give the basics of symmetrized, perturbed hyperbolic tangent neural network deterministic real operators, see also [10].

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In Section3, we give the basics of symmetrized, perturbed generalized logistic neural network deterministic real operators, see also [11].

In Section 4, we present the basics of fuzzy real analysis and we introduce the symmetrized and perturbed fuzzy neural network operators corresponding to Sections 2, 3. We prove also some very important properties. See also [5].

In section 5, we demonstrate our fuzzy results: fuzzy symmetrized and perturbed neural network quantitative approximations, the fuzzy analogs of the Sections 2, 3 positive linear operators generated results.

Our fuzzy feed-forward neural networks (FFNNs) are with one hidden layer. About neural networks in general study [15], [17], [18].

2. **Background I: About symmetrized perturbed hyperbolic tangent Neural Network Operators** ([10])

Here we follow [8], pp. 455-460.

Our activation function here to be used is

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - q e^{-\lambda x}}{e^{\lambda x} + q e^{-\lambda x}}, \quad \lambda, q > 0, \ x \in \mathbb{R}. \tag{1}$$

Above λ is the parameter and q is the deformation coefficient, typically it is $0 < \lambda, q \leq 1$.

For more read Chapter 18 of [8]: "q-Deformed and λ-Parametrized Hyperbolic Tangent based Banach space Valued Ordinary and Fractional Neural Network Approximation".

This chapter motivates our current work.

The proposed "symmetrization method" aims to use half data feed to our neural networks.

We will employ the following density function

$$M_{q.\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \tag{2}$$

$\forall x \in \mathbb{R}$; $q, \lambda > 0$.

We have that

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \ \ \forall \ x \in \mathbb{R}; \ q, \lambda > 0, \tag{3}$$

and

$$M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x), \ \ \forall \ x \in \mathbb{R}; \ q, \lambda > 0. \tag{4}$$

Adding (3) and (4) we obtain

$$M_{q,\lambda}(-x) + M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x), \tag{5}$$

the key to this work.

So that

$$\Phi(x) := \frac{M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x)}{2} > 0, \tag{6}$$

is an even function, symmetric with respect to the y-axis.

By (18.18) of [8], we have

$$\begin{aligned} & M_{q,\lambda}\left(\tfrac{\ln q}{2\lambda}\right) = \tfrac{\tanh(\lambda)}{2}, \\ & \text{and} \\ & M_{\frac{1}{q},\lambda}\left(-\tfrac{\ln q}{2\lambda}\right) = \tfrac{\tanh(\lambda)}{2}, \quad \lambda > 0. \end{aligned} \tag{7}$$

sharing the same maximum at symmetric points.

By Theorem 18.1, p. 458 of [8], we have that

$$\begin{aligned} & \sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \ \ \forall\, x \in \mathbb{R},\ \lambda, q > 0, \\ & \text{and} \\ & \sum_{i=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x-i) = 1, \ \ \forall\, x \in \mathbb{R},\ \lambda, q > 0. \end{aligned} \tag{8}$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} \Phi(x-i) = 1, \ \ \forall\, x \in \mathbb{R}. \tag{9}$$

By Theorem 18.2, p. 459 of [8], we have that

$$\begin{aligned} & \int_{-\infty}^{\infty} M_{q,\lambda}(x)\,dx = 1, \\ & \text{and} \\ & \int_{-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x)\,dx = 1, \end{aligned} \tag{10}$$

so that

$$\int_{-\infty}^{\infty} \Phi(x)\,dx = 1, \tag{11}$$

therefore Φ is a density function.

By Theorem 18.3, p. 459 of [8], we have:

Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q, \lambda > 0$. Then

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} M_{q,\lambda}(nx-k) < 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = T e^{-2\lambda n^{(1-\alpha)}}, \tag{12}$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Similarly, we get that

$$\sum_{k=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(nx-k) < T e^{-2\lambda n^{(1-\alpha)}}. \tag{13}$$

Consequently we obtain that

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{array}\right.}^{\infty} \Phi\left(nx-k\right) < Te^{-2\lambda n^{(1-\alpha)}}, \tag{14}$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Here $\lceil\cdot\rceil$ denotes the ceiling of the number, and $\lfloor\cdot\rfloor$ its integral part.

We mention

Theorem 18.4 (p. 459, [8]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leq \lfloor nb\rfloor$. For $q, \lambda > 0$, we consider $\lambda_q > z_0 > 0$, such that $M_{q,\lambda}(z_0) = M_{q,\lambda}(0)$, and $\lambda_q > 1$. Then

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right)} < \max\left\{\frac{1}{M_{q,\lambda}\left(\lambda_q\right)}, \frac{1}{M_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Delta\left(q\right). \tag{15}$$

Similarly, we consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $M_{\frac{1}{q},\lambda}(z_1) = M_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$. Thus

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{\frac{1}{q},\lambda}\left(nx-k\right)} < \max\left\{\frac{1}{M_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}, \frac{1}{M_{q,\lambda}\left(\lambda_q\right)}\right\} = \Delta\left(q\right). \tag{16}$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right) > \frac{1}{\Delta\left(q\right)}, \tag{17}$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{\frac{1}{q},\lambda}\left(nx-k\right) > \frac{1}{\Delta\left(q\right)}. \tag{18}$$

Consequently it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(M_{q,\lambda}\left(nx-k\right) + M_{\frac{1}{q},\lambda}\left(nx-k\right)\right)}{2} > \frac{2}{2\Delta\left(q\right)} = \frac{1}{\Delta\left(q\right)}, \tag{19}$$

so that

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(M_{q,\lambda}(nx-k) + M_{\frac{1}{q},\lambda}(nx-k)\right)}{2}} < \Delta\left(q\right), \tag{20}$$

that is

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)}<\Delta\left(q\right). \tag{21}$$

We have proved

Theorem 2.1. *Let $x\in[a,b]\subset\mathbb{R}$ and $n\in\mathbb{N}$ so that $\lceil na\rceil\leq\lfloor nb\rfloor$. For $q,\lambda>0$, we consider $\lambda_q>z_0>0$, such that $M_{q,\lambda}\left(z_0\right)=M_{q,\lambda}\left(0\right)$, and $\lambda_q>1$. Also consider $\lambda_{\frac{1}{q}}>z_1>0$, such that $M_{\frac{1}{q},\lambda}\left(z_1\right)=M_{\frac{1}{q},\lambda}\left(0\right)$, and $\lambda_{\frac{1}{q}}>1$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)}<\Delta\left(q\right). \tag{22}$$

We make

Remark 2.2. I) By Remark 18.5, p. 460 of [8], we have

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}\left(nx_1-k\right)\neq1,\ \text{for some } x_1\in[a,b], \tag{23}$$

and

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{\frac{1}{q},\lambda}\left(nx_2-k\right)\neq1,\ \text{for some } x_2\in[a,b]. \tag{24}$$

Therefore it holds

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\left(M_{q,\lambda}\left(nx_1-k\right)+M_{\frac{1}{q},\lambda}\left(nx_2-k\right)\right)}{2}\neq1. \tag{25}$$

Hence

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\left(M_{q,\lambda}\left(nx_1-k\right)+M_{\frac{1}{q},\lambda}\left(nx_1-k\right)\right)}{2}\neq1, \tag{26}$$

even if

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{\frac{1}{q},\lambda}\left(nx_1-k\right)=1, \tag{27}$$

because then

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{M_{q,\lambda}\left(nx_1-k\right)}{2}+\frac{1}{2}\neq1, \tag{28}$$

equivalently

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{M_{q,\lambda}(nx_1-k)}{2}\neq\frac{1}{2}, \tag{29}$$

true by

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx_1-k)\neq 1. \tag{30}$$

II) Let $[a,b]\subset\mathbb{R}$. For large n we always have $\lceil na\rceil\leq\lfloor nb\rfloor$. Also $a\leq\frac{k}{n}\leq b$, iff $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$. So in general it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\leq 1. \tag{31}$$

We need

Definition 2.3. Let $f\in C([a,b])$, $x\in[a,b]$ and $n\in\mathbb{N}:\lceil na\rceil\leq\lfloor nb\rfloor$. We introduce and define the real valued symmetrized and perturbed hyperbolic tangent positive linear neural network operators

$$H_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Phi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)}=$$

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\left(M_{q,\lambda}(nx-k)+M_{\frac{1}{q},\lambda}(nx-k)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(M_{q,\lambda}(nx-k)+M_{\frac{1}{q},\lambda}(nx-k)\right)}. \tag{32}$$

Infact, it is $H_n(f)\in C([a,b])$; and $H_n(1)=1$.

The modulus of continuity is defined by

$$\omega_1(f,\delta):=\sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}}|f(x)-f(y)|,\quad\delta>0. \tag{33}$$

In this work $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$; where $\|\cdot\|_\infty$ is the supremum norm.

The following approximation results, all from [10], are valid.

Theorem 2.4. *Let $f\in C([a,b])$. Then*

$$\|H_n(f)-f\|_\infty\leq 2\omega_1\left(f,\sqrt{\Delta(q)}\sqrt{\frac{1}{n^{2\alpha}}+(b-a)^2Te^{-2\lambda n^{(1-\alpha)}}}\right)\to 0,\quad \textit{as } n\to+\infty. \tag{34}$$

So that $\lim_{n\to\infty} H_n(f) = f$*, uniformly.*

It holds

Theorem 2.5. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous and* 2π *periodic function with modulus of continuity* ω_1*. Here* $\|\cdot\|_\infty$ *denotes the sup-norm over* $[a,b] \subset \mathbb{R}$*, and the operators* H_n *are acting on such* f *over* $[a,b]$*;* $n \in \mathbb{N} : n^{1-\alpha} > 2$*,* $0 < \alpha < 1$*. Then*

$$\|H_n(f) - f\|_\infty \le 2\omega_1 \left(f, \pi\sqrt{\Delta(q)} \sqrt{\omega_1\left(\sin^2, \frac{1}{2n^\alpha}\right) + Te^{-2\lambda n^{(1-\alpha)}}} \right). \tag{35}$$

We also mention, all from [10].

Theorem 2.6. *Denote* $(N \in \mathbb{N})$

$$F_N := \left\| \left(H_n \left(|t-\cdot|^N \right) \right)(\cdot) \right\|_\infty^{\frac{1}{N}} < +\infty, \tag{36}$$

where $\|\cdot\|_\infty$ *is the supremum norm.*

Let $f \in C^N([a,b])$*. Then*

$$\|H_n f - f\|_\infty \le \sum_{\overline{k}=1}^{N} \frac{\left\|f^{(\overline{k})}\right\|_\infty}{\overline{k}!} \left\| \left(H_n (t-\cdot)^{\overline{k}} \right)(\cdot) \right\|_\infty + \tag{37}$$

$$\omega_1\left(f^{(N)}, F_N\right) F_N^{N-1} \left(\frac{(b-a)}{(N+1)!} + \frac{F_N}{2N!} + \frac{F_N^2}{8(b-a)(N-1)!} \right), \quad \forall\, n \in \mathbb{N}.$$

Furthermore it holds,

$$F_N \le \sqrt[N]{\Delta(q)} \sqrt[N]{\frac{1}{n^{N\alpha}} + (b-a)^N Te^{-2\lambda n^{(1-\alpha)}}}, \tag{38}$$

and for $\overline{k} = 1, ..., N$*, we have, similarly, that*

$$\left\| \left(H_n (t-\cdot)^{\overline{k}} \right)(\cdot) \right\|_\infty \le \Delta(q) \left\{ \frac{1}{n^{\overline{k}\alpha}} + (b-a)^{\overline{k}} Te^{-2\lambda n^{(1-\alpha)}} \right\}. \tag{39}$$

Corollary 2.7. Here $F_1 := \|(H_n(|t-\cdot|))(\cdot)\|_\infty < \infty$. Let $f \in C^1([a,b])$. Then

$$\|H_n f - f\|_\infty \le \|f'\|_\infty \|(H_n(t-\cdot))(\cdot)\|_\infty + \tag{40}$$

$$\frac{1}{2}\omega_1(f', F_1) \left((b-a) + F_1 + \frac{F_1^2}{4(b-a)} \right).$$

Here it is

$$F_1 \le \Delta(q) \left(\frac{1}{n^\alpha} + (b-a) Te^{-2\lambda n^{(1-\alpha)}} \right). \tag{41}$$

Corollary 2.8. Here $F_2 := \left\| \left(H_n (t-\cdot)^2 \right)(\cdot) \right\|_\infty^{\frac{1}{2}} < +\infty$. Let $f \in C^2([a,b])$. Then

$$\|H_n f - f\|_\infty \le \|f'\|_\infty \|(H_n(t-\cdot))(\cdot)\|_\infty + \frac{\|f''\|_\infty}{2} \left\| \left(H_n (t-\cdot)^2 \right)(\cdot) \right\|_\infty \tag{42}$$

$$+\frac{1}{2}\omega_1(f'', F_2) F_2 \left(\frac{(b-a)}{3} + \frac{F_2}{2} + \frac{F_2^2}{4(b-a)} \right).$$

Here they are

$$F_2 \leq \sqrt{\Delta(q)}\sqrt{\frac{1}{n^{2\alpha}} + (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}}}, \tag{43}$$

and

$$\left\| \left(H_n (t-\cdot)^2 \right)(\cdot) \right\|_\infty \leq \Delta(q) \left(\frac{1}{n^{2\alpha}} + (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}} \right). \tag{44}$$

Theorem 2.9. *Let* $\varphi := \left\| \left(H_n (t-\cdot)^2 \right)(\cdot) \right\|_\infty^{\frac{1}{2}} < +\infty$, *and* $r > 0$. *Let also* $f \in C^1([a,b])$. *Then*

$$\|H_n f - f\|_\infty - \|f'\|_\infty \|(H_n(t-\cdot))(\cdot)\|_\infty \leq \tag{45}$$

$$\begin{cases} \frac{1}{8r}(2+r)^2 \omega_1(f', r\varphi)\varphi, & \text{if } r \leq 2; \\ \\ \omega_1(f', r\varphi)\varphi, & \text{if } r > 2. \end{cases}$$

We derive the following useful estimates:

Corollary 2.10. For $\overline{k} = 1, ..., N \in \mathbb{N}$, we denote by

$$\psi_{1,\overline{k},n} := \Delta(q) \left\{ \frac{1}{n^{\overline{k}\alpha}} + (b-a)^{\overline{k}} T e^{-2\lambda n^{(1-\alpha)}} \right\}, \tag{46}$$

and $\|\cdot\|_\infty$ is the supremum norm.

Let $f \in C^N([a,b])$. Then

$$\|H_n f - f\|_\infty \leq \sum_{\overline{k}=1}^{N} \frac{\left\| f^{(\overline{k})} \right\|_\infty}{\overline{k}!} \psi_{1,\overline{k},n} + \tag{47}$$

$$\omega_1\left(f^{(N)}, \psi_{1,N,n}^{\frac{1}{N}} \right) \psi_{1,N,n}^{\frac{N-1}{N}} \left(\frac{(b-a)}{(N+1)!} + \frac{\psi_{1,N,n}^{\frac{1}{N}}}{2N!} + \frac{\psi_{1,N,n}^{\frac{2}{N}}}{8(b-a)(N-1)!} \right), \quad \forall\, n \in \mathbb{N}.$$

Proof. By Theorem 2.6. □

Corollary 2.11. Let $f \in C^1([a,b])$. Then

$$\|H_n f - f\|_\infty \leq \|f'\|_\infty \psi_{1,1,n} +$$

$$\frac{1}{2}\omega_1(f', \psi_{1,1,n}) \left((b-a) + \psi_{1,1,n} + \frac{\psi_{1,1,n}^2}{4(b-a)} \right). \tag{48}$$

Proof. By Corollary 2.7 and (46). □

Corollary 2.12. Let $f \in C^2([a,b])$. Then

$$\|H_n f - f\|_\infty \leq \|f'\|_\infty \psi_{1,1,n} + \frac{\|f''\|_\infty}{2} \psi_{1,2,n}$$

$$+\frac{1}{2}\omega_1\left(f'', \psi_{1,2,n}^{\frac{1}{2}} \right) \psi_{1,2,n}^{\frac{1}{2}} \left(\frac{(b-a)}{3} + \frac{\psi_{1,2,n}^{\frac{1}{2}}}{2} + \frac{\psi_{1,2,n}}{4(b-a)} \right), \ \forall\, n \in \mathbb{N}. \tag{49}$$

Proof. By Corollary 2.8 and (46). □

Corollary 2.13. Let $f \in C^1([a,b])$, $r > 0$. Then

$$\|H_n f - f\|_\infty \leq \|f'\|_\infty \psi_{1,1,n} +$$

$$\begin{cases} \frac{1}{8r}(2+r)^2 \omega_1\left(f', r\psi_{1,2,n}^{\frac{1}{2}}\right)\psi_{1,2,n}^{\frac{1}{2}}, & \text{if } r \leq 2; \\ \\ \omega_1\left(f', r\psi_{1,2,n}^{\frac{1}{2}}\right)\psi_{1,2,n}^{\frac{1}{2}}, & \text{if } r > 2. \end{cases} \tag{50}$$

Proof. By Theorem 2.9 and (46). □

3. Background II: About symmetrized, perturbed generalized logistic Neural Network Operators, [11]

Here we follow [8], pp. 395-417.

Our activation function here to be used is the q-deformed and λ-parametrized function

$$\varphi_{q,\lambda}(x) = \frac{1}{1 + qA^{-\lambda x}}, \quad x \in \mathbb{R},\ q, \lambda > 0,\ A > 1. \tag{51}$$

This is the A-generalized logistic function.

For more read Chapter 16 of [8]: "Banach space valued ordinary and fractional neural network approximation based on q-deformed and λ-parametrized A-generalized logistic function".

This chapter motivates our current work.

The proposed "symmetrization technique" aims to use half data feed to our neural networks.

We will employ the following density function

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R},\ q, \lambda > 0. \tag{52}$$

We have that

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \tag{53}$$

and

$$G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x), \ \forall\, x \in \mathbb{R}. \tag{54}$$

Adding (3) and (4) we obtain

$$G_{q,\lambda}(-x) + G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x), \ \forall\, x \in \mathbb{R}, \tag{55}$$

the key to this work.

So that

$$W(x) := \frac{G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x)}{2} > 0, \tag{56}$$

is an even function, symmetric with respect to the y-axis.

The global maximum of $G_{q,\lambda}$ is given by (16.18), p. 401 of [8] as

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{A^\lambda - 1}{2(A^\lambda + 1)}. \tag{57}$$

And, the global max of $G_{\frac{1}{q},\lambda}$ is

$$G_{\frac{1}{q},\lambda}\left(\frac{\log_A \frac{1}{q}}{\lambda}\right) = G_{\frac{1}{q},\lambda}\left(\frac{-\log_A q}{\lambda}\right) = \frac{A^\lambda - 1}{2(A^\lambda + 1)}, \tag{58}$$

both sharing the same maximum at symmetric points.

By Theorem 16.1, p. 401 of [8], we have that

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(x - i) = 1, \ \forall\, x \in \mathbb{R}, \lambda, q > 0, \ A > 1, \tag{59}$$

and

$$\sum_{i=-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x - i) = 1, \ \forall\, x \in \mathbb{R}, \lambda, q > 0, \ A > 1. \tag{60}$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} W(x - i) = 1, \ \forall\, x \in \mathbb{R}. \tag{61}$$

By Theorem 16.2, p. 402 of [8], we have that

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\, dx = 1, \quad \lambda, q > 0, \ A > 1, \tag{62}$$

similarly it holds

$$\int_{-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x)\, dx = 1, \tag{63}$$

so that

$$\int_{-\infty}^{\infty} W(x)\, dx = 1, \tag{64}$$

therefore W is a density function.

By Theorem 16.3, p. 402 of [8], we have:

Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} G_{q,\lambda}(nx - k) < 2 \max\left\{q, \frac{1}{q}\right\} \frac{1}{A^{\lambda(n^{1-\alpha}-2)}} = \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \tag{65}$$

where $\lambda, q > 0,\ A > 1;\ \gamma := 2\max\left\{q, \frac{1}{q}\right\}$.

Similarly, we get that

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} G_{\frac{1}{q},\lambda}(nx-k) < \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}. \tag{66}$$

Consequently we obtain that

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} W(nx-k) < \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \tag{67}$$

where $\gamma := 2\max\left\{q, \frac{1}{q}\right\}$.

We mention

Theorem 16.4 (p. 402, [8]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0$, $\lambda > 0$, $A > 1$, we consider the number $\lambda_q > z_0 > 0$ with $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$, and $\lambda_q > 1$. Then

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx-k)} < \max\left\{\frac{1}{G_{q,\lambda}(\lambda_q)}, \frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: K(q). \tag{68}$$

Similarly, we consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $G_{\frac{1}{q},\lambda}(z_1) = G_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$. Thus

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{\frac{1}{q},\lambda}(nx-k)} < \max\left\{\frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}, \frac{1}{G_{q,\lambda}(\lambda_q)}\right\} = K(q). \tag{69}$$

Hence

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx-k) > \frac{1}{K(q)}, \tag{70}$$

and

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{\frac{1}{q},\lambda}(nx-k) > \frac{1}{K(q)}. \tag{71}$$

Consequently it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{\left(G_{q,\lambda}(nx-k) + G_{\frac{1}{q},\lambda}(nx-k)\right)}{2} > \frac{2}{2K(q)} = \frac{1}{K(q)}, \tag{72}$$

so that

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\left(G_{q,\lambda}(nx-k)+G_{\frac{1}{q},\lambda}(nx-k)\right)}{2}} < K\left(q\right), \tag{73}$$

that is

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}W\left(nx-k\right)} < K\left(q\right). \tag{74}$$

We have proved

Theorem 3.1. *Let $x\in[a,b]\subset\mathbb{R}$ and $n\in\mathbb{N}$ so that $\lceil na\rceil\leq\lfloor nb\rfloor$. For $q,\lambda>0$, $A>1$, we consider $\lambda_q>z_0>0$ with $G_{q,\lambda}\left(z_0\right)=G_{q,\lambda}\left(0\right)$, and $\lambda_q>1$. Also consider $\lambda_{\frac{1}{q}}>z_1>0$, such that $G_{\frac{1}{q},\lambda}\left(z_1\right)=G_{\frac{1}{q},\lambda}\left(0\right)$, and $\lambda_{\frac{1}{q}}>1$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}W\left(nx-k\right)} < K\left(q\right). \tag{75}$$

We make

Remark 3.2. I) By Remark 16.5, p. 402 of [8], we have

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(nx_1-k\right)\neq1,\ \text{for some } x_1\in[a,b], \tag{76}$$

and

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{\frac{1}{q},\lambda}\left(nx_2-k\right)\neq1,\ \text{for some } x_2\in[a,b]. \tag{77}$$

Therefore it holds

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\left(G_{q,\lambda}\left(nx_1-k\right)+G_{\frac{1}{q},\lambda}\left(nx_2-k\right)\right)}{2}\neq1. \tag{78}$$

Hence it is

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\left(G_{q,\lambda}\left(nx_1-k\right)+G_{\frac{1}{q},\lambda}\left(nx_1-k\right)\right)}{2}\neq1, \tag{79}$$

even if

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{\frac{1}{q},\lambda}\left(nx_1-k\right)=1, \tag{80}$$

because then

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{G_{q,\lambda}\left(nx_1-k\right)}{2}+\frac{1}{2}\neq 1, \tag{81}$$

equivalently

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{G_{q,\lambda}\left(nx_1-k\right)}{2}\neq\frac{1}{2}, \tag{82}$$

true by

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(nx_1-k\right)\neq 1. \tag{83}$$

II) Let $[a,b]\subset\mathbb{R}$. For large n we always have $\lceil na\rceil\leq\lfloor nb\rfloor$. Also $a\leq\frac{k}{n}\leq b$, iff $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$. So in general it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}W\left(nx-k\right)\leq 1. \tag{84}$$

We need:

Definition 3.3. Let $f\in C\left([a,b]\right)$ and $n\in\mathbb{N}:\lceil na\rceil\leq\lfloor nb\rfloor$. We introduce and define the real valued symmetrized and perturbed generalized logistic positive linear neural network operators

$$S_n^s\left(f,x\right):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)W\left(nx-k\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}W\left(nx-k\right)}=$$

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\left(G_{q,\lambda}\left(nx-k\right)+G_{\frac{1}{q},\lambda}\left(nx-k\right)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(G_{q,\lambda}\left(nx-k\right)+G_{\frac{1}{q},\lambda}\left(nx-k\right)\right)}. \tag{85}$$

Infact, it is $S_n^s\left(f\right)\in C\left([a,b]\right)$; and $S_n^s\left(1\right)=1$.

The modulus of continuity is again defined by (33).

The same ω_1 is defined for $f\in C_{uB}\left(\mathbb{R}\right)$ (uniformly continuous and bounded functions) and for $f\in C_B\left(\mathbb{R}\right)$ (bounded and continuous functions) and for $f\in C_u\left(\mathbb{R}\right)$ (uniformly continuous functions).

The fact $f\in C\left([a,b]\right)$ or $f\in C_u\left(\mathbb{R}\right)$, is equivalent to $\lim_{\delta\to 0}\omega_1\left(f,\delta\right)=0$.

The following approximation results, all from [11], are valid.

Theorem 3.4. *Let $f \in C([a,b])$. Then*

(86)
$$\left\|S_n^s(f) - f\right\|_\infty \leq 2\omega_1\left(f, \sqrt{K(q)}\sqrt{\frac{1}{n^{2\alpha}} + (b-a)^2 \gamma A^{-\lambda(n^{1-\alpha}-2)}}\right) \to 0, \quad as\ n \to +\infty.$$

So that $\lim_{n\to\infty} S_n^s(f) = f$, *uniformly.*

It holds

Theorem 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and 2π periodic function with modulus of continuity ω_1. Here $\|\cdot\|_\infty$ denotes the sup-norm over $[a,b] \subset \mathbb{R}$, and the operators S_n^s are acting on such f over $[a,b]$; $n \in \mathbb{N} : n^{1-\alpha} > 2$, $0 < \alpha < 1$. Then*

$$\left\|S_n^s(f) - f\right\|_\infty \leq 2\omega_1\left(f, \pi\sqrt{K(q)}\sqrt{\omega_1\left(\sin^2, \frac{1}{2n^\alpha}\right) + \gamma A^{-\lambda(n^{1-\alpha}-2)}}\right). \tag{87}$$

Also from [11] come:

Theorem 3.6. *Denote* ($N \in \mathbb{N}$)

$$\widetilde{F}_N := \left\|\left(S_n^s\left(|t-\cdot|^N\right)\right)(\cdot)\right\|_\infty^{\frac{1}{N}} < +\infty, \tag{88}$$

where $\|\cdot\|_\infty$ is the supremum norm.

Let $f \in C^N([a,b])$. Then

$$\left\|S_n^s f - f\right\|_\infty \leq \sum_{\overline{k}=1}^{N} \frac{\left\|f^{(\overline{k})}\right\|_\infty}{\overline{k}!}\left\|\left(S_n^s(t-\cdot)^{\overline{k}}\right)(\cdot)\right\|_\infty + \tag{89}$$
$$\omega_1\left(f^{(N)}, \widetilde{F}_N\right)\widetilde{F}_N^{N-1}\left(\frac{(b-a)}{(N+1)!} + \frac{\widetilde{F}_N}{2N!} + \frac{\widetilde{F}_N^2}{8(b-a)(N-1)!}\right), \quad \forall\, n \in \mathbb{N}.$$

Furthermore it holds,

$$\widetilde{F}_N \leq \sqrt[N]{K(q)}\sqrt[N]{\frac{1}{n^{N\alpha}} + (b-a)^N \gamma A^{-\lambda(n^{1-\alpha}-2)}}, \tag{90}$$

and for $\overline{k} = 1, ..., N$, we have, similarly, that

$$\left\|\left(S_n^s(t-\cdot)^{\overline{k}}\right)(\cdot)\right\|_\infty \leq K(q)\left\{\frac{1}{n^{\overline{k}\alpha}} + (b-a)^{\overline{k}}\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}\right\}. \tag{91}$$

Corollary 3.7. Here $\widetilde{F}_1 := \|(S_n^s(|t-\cdot|))(\cdot)\|_\infty < \infty$. Let $f \in C^1([a,b])$. Then

$$\left\|S_n^s f - f\right\|_\infty \leq \|f'\|_\infty \|(S_n^s(t-\cdot))(\cdot)\|_\infty + \tag{92}$$
$$\frac{1}{2}\omega_1\left(f', \widetilde{F}_1\right)\left((b-a) + \widetilde{F}_1 + \frac{\widetilde{F}_1^2}{4(b-a)}\right).$$

Here it is

$$\widetilde{F}_1 \leq K(q)\left(\frac{1}{n^\alpha} + (b-a)\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}\right). \tag{93}$$

Corollary 3.8. Here $\widetilde{F}_2 := \left\| \left(S_n^s (t-\cdot)^2 \right)(\cdot) \right\|_\infty^{\frac{1}{2}} < +\infty$. Let $f \in C^2([a,b])$. Then

$$\|S_n^s f - f\|_\infty \leq \|f'\|_\infty \|(S_n^s (t-\cdot))(\cdot)\|_\infty + \frac{\|f''\|_\infty}{2} \left\| \left(S_n^s (t-\cdot)^2 \right)(\cdot) \right\|_\infty \tag{94}$$

$$+ \frac{1}{2} \omega_1 \left(f'', \widetilde{F}_2 \right) \widetilde{F}_2 \left(\frac{(b-a)}{3} + \frac{\widetilde{F}_2}{2} + \frac{\widetilde{F}_2^2}{4(b-a)} \right).$$

Here they are

$$\widetilde{F}_2 \leq \sqrt{K(q)} \sqrt{\frac{1}{n^{2\alpha}} + (b-a)^2 \gamma A^{-\lambda(n^{1-\alpha}-2)}}, \tag{95}$$

and

$$\left\| \left(S_n^s (t-\cdot)^2 \right)(\cdot) \right\|_\infty \leq K(q) \left(\frac{1}{n^{2\alpha}} + (b-a)^2 \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)} \right). \tag{96}$$

Theorem 3.9. *Let* $\widetilde{\varphi} := \left\| \left(S_n^s (t-\cdot)^2 \right)(\cdot) \right\|_\infty^{\frac{1}{2}} < +\infty$*, and* $r > 0$*. Let also* $f \in C^1([a,b])$*. Then*

$$\|S_n^s f - f\|_\infty - \|f'\|_\infty \|(S_n^s (t-\cdot))(\cdot)\|_\infty \leq \tag{97}$$

$$\begin{cases} \frac{1}{8r} (2+r)^2 \omega_1 (f', r\widetilde{\varphi}) \widetilde{\varphi}, & \text{if } r \leq 2; \\ \\ \omega_1 (f', r\widetilde{\varphi}) \widetilde{\varphi}, & \text{if } r > 2. \end{cases}$$

We derive the following useful estimates:

Corollary 3.10. For $\overline{k} = 1, ..., N \in \mathbb{N}$, we denote by

$$\psi_{2,\overline{k},n} := K(q) \left\{ \frac{1}{n^{\overline{k}\alpha}} + (b-a)^{\overline{k}} \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)} \right\}. \tag{98}$$

Let $f \in C^N([a,b])$. Then

$$\|S_n^s f - f\|_\infty \leq \sum_{\overline{k}=1}^{N} \frac{\left\| f^{(\overline{k})} \right\|_\infty}{\overline{k}!} \psi_{2,\overline{k},n} +$$

$$\omega_1 \left(f^{(N)}, \psi_{2,N,n}^{\frac{1}{N}} \right) \psi_{2,N,n}^{\frac{N-1}{N}} \left(\frac{(b-a)}{(N+1)!} + \frac{\psi_{2,N,n}^{\frac{1}{N}}}{2N!} + \frac{\psi_{2,N,n}^{\frac{2}{N}}}{8(b-a)(N-1)!} \right), \quad \forall\, n \in \mathbb{N}. \tag{99}$$

Proof. By Theorem 3.6. □

Corollary 3.11. Let $f \in C^1([a,b])$. Then

$$\|S_n^s f - f\|_\infty \leq \|f'\|_\infty \psi_{2,1,n} + \tag{100}$$

$$\frac{1}{2} \omega_1 (f', \psi_{2,1,n}) \left((b-a) + \psi_{2,1,n} + \frac{\psi_{2,1,n}^2}{4(b-a)} \right).$$

Proof. By Corollary 3.7 and (98). □

Corollary 3.12. Let $f \in C^2([a,b])$. Then

$$\|S_n^s f - f\|_\infty \leq \|f'\|_\infty \psi_{2,1,n} + \frac{\|f''\|_\infty}{2} \psi_{2,2,n}$$

$$+\frac{1}{2}\omega_1\left(f'', \psi_{2,2,n}^{\frac{1}{2}}\right)\psi_{2,2,n}^{\frac{1}{2}}\left(\frac{(b-a)}{3} + \frac{\psi_{2,2,n}^{\frac{1}{2}}}{2} + \frac{\psi_{2,2,n}}{4(b-a)}\right), \quad \forall\, n \in \mathbb{N}. \tag{101}$$

Proof. By Corollary 3.8 and (98). □

Corollary 3.13. Let $f \in C^1([a,b])$, $r > 0$. Then

$$\|S_n^s f - f\|_\infty \leq \|f'\|_\infty \psi_{2,1,n} +$$

$$\begin{cases} \frac{1}{8r}(2+r)^2 \omega_1\left(f', r\psi_{2,2,n}^{\frac{1}{2}}\right)\psi_{2,2,n}^{\frac{1}{2}}, & \text{if } r \leq 2; \\ \omega_1\left(f', r\psi_{2,2,n}^{\frac{1}{2}}\right)\psi_{2,2,n}^{\frac{1}{2}}, & \text{if } r > 2. \end{cases} \tag{102}$$

Proof. By Theorem 3.9 and (98). □

4. **Background III: Fuzzy Mathematical Analysis**

We need the following basic background

Definition 4.1. (see [20]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:

(i) is normal, i.e., $\exists\, x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\, x, y \in \mathbb{R}$, $\forall\, \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e. $\forall\, x_0 \in \mathbb{R}$ and $\forall\, \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\, x \in V(x_0)$.

(iv) The set $\overline{\text{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\text{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval on $\mathbb{R}$ ([14]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \forall\, r \in [0,1],$$

where

$[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and

$\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [20]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \ \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\},$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \ u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [20], [21].

Here $\sum^*$ stands for fuzzy summation and $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neural element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \forall\, u \in \mathbb{R}_{\mathcal{F}}.$$

Denote

$$D^*(f,g) = \sup_{x \in X \subseteq \mathbb{R}} D(f(x), g(x)),$$

where $f, g : X \to \mathbb{R}_{\mathcal{F}}$.

We mention

Definition 4.2. Let $f : X \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, X interval, we define the (first) fuzzy modulus of continuity of f by

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{x,y \in X,\ |x-y| \leq \delta} D(f(x), f(y)), \ \delta > 0. \tag{103}$$

When $g : X \subseteq \mathbb{R} \to \mathbb{R}$, we define

$$\omega_1(g,\delta) = \omega_1(g,\delta)_X = \sup_{x,y \in X,\ |x-y| \leq \delta} |g(x) - g(y)|.$$

We define by $C_{\mathcal{F}}^{U}(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 4.3. ([4]) Let $f \in C_{\mathcal{F}}^{U}(X)$. Then $\omega_1^{(\mathcal{F})}(f, \delta)_X < \infty$, for any $\delta > 0$.

By [5], p. 129 we have that $C_{\mathcal{F}}^{U}([a,b]) = C_{\mathcal{F}}([a,b])$, fuzzy continuous functions on $[a,b] \subset \mathbb{R}$.

Proposition 4.4. ([4]) It holds

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})}(f, \delta)_X = \omega_1^{(\mathcal{F})}(f, 0)_X = 0,$$

iff $f \in C_{\mathcal{F}}^{U}(X)$, where X is a compact interval.

Proposition 4.5. ([4]) Here $[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right]$, $r \in [0,1]$. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.

Note 4.6. It is clear by Propositions 4.4, 4.5, that if $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$, then $f_{\pm}^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$). Also if $f \in C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ implies $f_{\pm}^{(r)} \in C_b(\mathbb{R})$ (continuous and bounded functions on $\mathbb{R}$).

Proposition 4.7. Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f, \delta)_X$, $\omega_1\left(f_-^{(r)}, \delta\right)_X$, $\omega_1\left(f_+^{(r)}, \delta\right)_X$ are finite for any $\delta > 0$, $r \in [0,1]$, where X any interval of $\mathbb{R}$.

Then

$$\omega_1^{(\mathcal{F})}(f, \delta)_X = \sup_{r \in [0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \delta\right)_X, \omega_1\left(f_+^{(r)}, \delta\right)_X\right\}. \tag{104}$$

Proof. Similar to Proposition 14.15, p. 246 of [5]. □

Definition 4.8. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.

Definition 4.9. ([19]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-differential at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{105}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$, similarly is defined $C_{\mathcal{F}}^{N}([a,b])$, $[a,b] \subset \mathbb{R}$.

We mention

Theorem 4.10. *([16]) Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in \mathbb{R}$, $0 \leq r \leq 1$. Clearly*

$$[f(t)]^{r} = \left[f(t)_{-}^{(r)}, f(t)_{+}^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_{\pm}^{(r)}$ are differentiable and

$$[f'(t)]^{r} = \left[\left(f(t)_{-}^{(r)} \right)', \left(f(t)_{+}^{(r)} \right)' \right].$$

I.e.

$$(f')_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)', \quad \forall\, r \in [0,1]. \tag{106}$$

Remark 4.11. ([3]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 4.10 we obtain

$$\left[f^{(i)}(t) \right]^{r} = \left[\left(f(t)_{-}^{(r)} \right)^{(i)}, \left(f(t)_{+}^{(r)} \right)^{(i)} \right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have that

$$\left(f^{(i)} \right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)^{(i)}, \tag{107}$$

for any $r \in [0,1]$, all $i = 0, 1, 2, ..., N$.

Note 4.12. ([3]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 4.10 we have $f_{\pm}^{(r)} \in C^{N}(\mathbb{R})$, for any $r \in [0,1]$.

Items 36-38 are valid also on $[a,b]$.

By [5], p. 131, if $f \in C_{\mathcal{F}}([a,b])$, then f is a fuzzy bounded function.

Remark 4.13. We try to determine and use ($W \subset \mathbb{R}$):

$$D^{*}(f, \widetilde{o}) = \sup_{x \in W} D(f(x), \widetilde{o}) =$$

$$\sup_{x \in W} \sup_{r \in [0,1]} \max \left\{ \left| f_{-}^{(r)}(x) \right|, \left| f_{+}^{(r)}(x) \right| \right\}.$$

By the principle of iterated suprema we find that

$$D^{*}(f, \widetilde{o}) = \sup_{r \in [0,1]} \max \left\{ \left\| f_{-}^{(r)} \right\|_{\infty}, \left\| f_{+}^{(r)} \right\|_{\infty} \right\}, \tag{108}$$

under the assumption $D^{*}(f, \widetilde{o}) < \infty$, that is f is assumed a fuzzy bounded function.

Above $\|\cdot\|_{\infty}$ is the supremum norm of the function over $W \subseteq \mathbb{R}$.

Another direct proof of (108) follows:

We easily see that

$$D^*(f,\widetilde{o}) \leq \sup_{r\in[0,1]} \max\left\{\left\|f_-^{(r)}\right\|_\infty, \left\|f_+^{(r)}\right\|_\infty\right\}.$$

On the other hand we observe that $\forall\ x \in W$: each

$$\left|f_\pm^{(r)}(x)\right| \leq \max\left\{|f_\pm^{(r)}(x)|\right\} \leq \sup_{r\in[0,1]} \max\left\{|f_\pm^{(r)}(x)|\right\} \leq$$

$$\sup_{x\in W} \sup_{r\in[0,1]} \max\left\{|f_\pm^{(r)}(x)|\right\} = D^*(f,\widetilde{o}).$$

That is, each

$$\left|f_\pm^{(r)}(x)\right| \leq D^*(f,\widetilde{o}),\ \ \forall\ x\in W, \tag{109}$$

hence each

$$\left\|f_\pm^{(r)}\right\|_\infty \leq D^*(f,\widetilde{o}), \tag{110}$$

and

$$\max\left\{\left\|f_\pm^{(r)}\right\|_\infty\right\} \leq D^*(f,\widetilde{o}), \tag{111}$$

and

$$\sup_{r\in[0,1]} \max\left\{\left\|f_-^{(r)}\right\|_\infty, \left\|f_+^{(r)}\right\|_\infty\right\} \leq D^*(f,\widetilde{o}), \tag{112}$$

proving (108).

The asumption $D^*(f,\widetilde{o}) < \infty$ implies $\left\|f_\pm^{(r)}\right\|_\infty < \infty$, $\forall\ r \in [0,1]$.

Clearly, it holds

$$\left|\left(f^{(j)}\right)_\pm^{(r)}(x)\right| \leq D\left(f^{(j)}(x),\widetilde{o}\right),\ \ j=1,...,N, \tag{113}$$

and

$$\left\|\left(f^{(N)}\right)_\pm^{(r)}\right\|_\infty \leq D^*\left(f^{(N)},\widetilde{o}\right). \tag{114}$$

We make

Definition 4.14. Let $f \in C_{\mathcal{F}}([a,b])$ (fuzzy continuous functions on $[a,b] \subset \mathbb{R}$), $n \in \mathbb{N}$. We define the following Fuzzy symmetrized and perturbed neural network operators

$$H_n^{\mathcal{F}}(f,x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{k}{n}\right) \odot \frac{\Phi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)}, \tag{115}$$

and

$$S_n^{s\mathcal{F}}(f,x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{k}{n}\right) \odot \frac{W(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} W(nx-k)}, \tag{116}$$

$\forall\ x \in [a,b]$, both are finite fuzzy sums.

Call

$$\Theta_1(x) := \Phi(x), \quad \Theta_2(x) := W(x), \tag{117}$$

and

$$L_{1n}^{\mathcal{F}}(f,x) := H_n^{\mathcal{F}}(f,x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot \frac{\Theta_1(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta_1(nx-k)}, \tag{118}$$

and

$$L_{2n}^{\mathcal{F}}(f,x) := S_n^{s\mathcal{F}}(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot \frac{\Theta_2(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta_2(nx-k)}, \tag{119}$$

$\forall\ x \in [a,b],\ n \in \mathbb{N}$.

Also, we call

$$L_{1n}(f,x) := H_n(f,x) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Theta_1(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta_1(nx-k)}, \tag{120}$$

and

$$L_{2n}(f,x) := S_n^{s}(f,x) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Theta_2(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta_2(nx-k)}, \tag{121}$$

$\forall\ x \in [a,b],\ n \in \mathbb{N}$.

We give the following important result:

Theorem 4.15. *Let $f \in C_{\mathcal{F}}([a,b])$. Then*

$$\left(L_{jn}^{\mathcal{F}}(f,x)\right)_{\pm}^{(r)} = L_{jn}\left(f_{\pm}^{(r)}, x\right), \tag{122}$$

respectively, $\forall\ r \in [0,1]$, $\forall\ x \in [a,b]$, $j = 1,2$, $n \in \mathbb{N}$.

Therefore we get

$$D\left(L_{jn}^{\mathcal{F}}(f,x), f(x)\right) =$$

$$\sup_{r \in [0,1]} \max\left\{\left|L_{jn}\left(f_{-}^{(r)}, x\right) - f_{-}^{(r)}(x)\right|, \left|L_{jn}\left(f_{+}^{(r)}, x\right) - f_{+}^{(r)}(x)\right|\right\}, \tag{123}$$

$\forall\ x \in [a,b],\ j = 1,2.$

Proof. Let $r \in [0,1]$, we observe that

$$\left[L_{jn}^{\mathcal{F}}(f,x)\right]^r = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left[f\left(\frac{k}{n}\right)\right]^r \left(\frac{\Theta_j(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta_j(nx-k)}\right) =$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left[f_-^{(r)}\left(\frac{k}{n}\right), f_+^{(r)}\left(\frac{k}{n}\right)\right]\left(\frac{\Theta_j(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta_j(nx-k)}\right) =$$

(124)

$$\left[\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f_-^{(r)}\left(\frac{k}{n}\right)\left(\frac{\Theta_j(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta_j(nx-k)}\right), \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f_+^{(r)}\left(\frac{k}{n}\right)\left(\frac{\Theta_j(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta_j(nx-k)}\right)\right] =$$

$$\left[L_{jn}\left(f_-^{(r)},x\right), L_{jn}\left(f_+^{(r)},x\right)\right], \quad \forall x\in[a,b],\ n\in\mathbb{N},$$

proving the claim. ☐

Definition 4.16. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. We call it fuzzy 2π periodic iff

$$f(x+2\pi) = f(x), \quad \forall\ x \in \mathbb{R}. \tag{125}$$

We have that ($r \in [0,1]$)

$$[f(x)]^r = [f(x+2\pi)]^r = \left[f_-^{(r)}(x+2\pi), f_+^{(r)}(x+2\pi)\right] = \left[f_-^{(r)}(x), f_+^{(r)}(x)\right]. \tag{126}$$

Hence $f_{\pm}^{(r)}$ are continuous and 2π periodic over $\mathbb{R}$, $\forall\ r \in [0,1]$.

5. **Main Results**

Here we present fuzzy symmetrized and perturbed positive linear operators generated neural network approximation results:

Theorem 5.1. *Let $f \in C_{\mathcal{F}}([a,b])$, $0<\alpha<1$, $x\in[a,b]$, $n\in\mathbb{N}$ with $n^{1-\alpha}>2$. Then*

1)

(127)

$$D\left(H_n^{\mathcal{F}}(f,x), f(x)\right) \le 2\omega_1^{(\mathcal{F})}\left(f, \sqrt{\Delta(q)}\sqrt{\frac{1}{n^{2\alpha}} + (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}}}\right) =: \rho_{1n},$$

$\forall\ x \in [a,b]$,

2)

$$D^*\left(H_n^{\mathcal{F}}(f), f\right) \le \rho_{1n}. \tag{128}$$

Thus $\lim_{n\to\infty}\left(H_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, $\lim_{n\to\infty} H_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, *pointwise and uniformly.*

Proof. We have that $f_{\pm}^{(r)} \in C([a,b])$, $\forall\, r \in [0,1]$. Hence by (34) we obtain

$$\left| H_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq$$

$$2\omega_1\left(f_{\pm}^{(r)}, \sqrt{\Delta(q)}\sqrt{\frac{1}{n^{2\alpha}} + (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}}}\right) \overset{(104)}{\leq}$$

$$2\omega_1^{(\mathcal{F})}\left(f, \sqrt{\Delta(q)}\sqrt{\frac{1}{n^{2\alpha}} + (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}}}\right). \tag{129}$$

Taking into account (123) the proof of the claim is completed. □

The counterpart of Theorem 5.1 follows.

Theorem 5.2. *All as in Theorem 5.1. Then*

1)

(130)

$$D\left(S_n^{s\mathcal{F}}(f,x), f(x)\right) \leq 2\omega_1^{(\mathcal{F})}\left(f, \sqrt{K(q)}\sqrt{\frac{1}{n^{2\alpha}} + (b-a)^2 \gamma A^{-\lambda(n^{1-\alpha}-2)}}\right) =: \rho_{2n},$$

$\forall\, x \in [a,b]$,

2)

$$D^*\left(S_n^{s\mathcal{F}}(f), f\right) \leq \rho_{2n}. \tag{131}$$

Thus $\lim_{n\to\infty}\left(S_n^{s\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, $\lim_{n\to\infty} S_n^{s\mathcal{F}}(f) \overset{D^*}{\to} f$, *pointwise and uniformly.*

Proof. Similar to Theorem 5.1. □

We continue with:

Theorem 5.3. *Let* $f \in C_{\mathcal{F}}(\mathbb{R})$ *be fuzzy* 2π *periodic with a modulus of continuity* $\omega_1^{(\mathcal{F})}$. *Here we consider* $f|_{[a,b]}$, $[a,b] \subset \mathbb{R}$, *and* $H_n^{\mathcal{F}}$ *acts on it;* $n \in \mathbb{N} : n^{1-\alpha} > 2$, $0 < \alpha < 1$. *Then*

1)

$$D\left(H_n^{\mathcal{F}}(f,x), f(x)\right) \leq$$

$$2\omega_1^{(\mathcal{F})}\left(f, \pi\sqrt{\Delta(q)}\sqrt{\omega_1\left(\sin^2, \frac{1}{2n^{2\alpha}}\right) + T e^{-2\lambda n^{(1-\alpha)}}}\right) =: \sigma_{1n}, \tag{132}$$

$\forall\, x \in [a,b]$,

2)

$$D^*\left(H_n^{\mathcal{F}}(f), f\right) \leq \sigma_{1n}. \tag{133}$$

Thus $\lim_{n\to\infty}\left(H_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, $\lim_{n\to\infty} H_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, *pointwise and uniformly.*

Proof. Similar to Theorem 5.1. □

Its counterpart follows:

Theorem 5.4. *Here all as in Theorem 5.3. Then*

1)

$$D\left(S_n^{s\mathcal{F}}(f,x), f(x)\right) \leq$$

$$2\omega_1^{(\mathcal{F})}\left(f, \pi\sqrt{K(q)}\sqrt{\omega_1\left(\sin^2, \frac{1}{2n^{2\alpha}}\right) + \gamma A^{-\lambda(n^{1-\alpha}-2)}}\right) =: \sigma_{2n}, \tag{134}$$

$\forall\ x \in [a,b]$,

2)

$$D^*\left(S_n^{s\mathcal{F}}(f), f\right) \leq \sigma_{2n}. \tag{135}$$

Thus $\lim_{n\to\infty}\left(S_n^{s\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, $\lim_{n\to\infty} S_n^{s\mathcal{F}}(f) \overset{D^*}{\to} f$, *pointwise and uniformly.*

Proof. Similar to Theorem 5.1. □

We continue with:

Theorem 5.5. *Here* $f \in C_{\mathcal{F}}^N([a,b])$, $N \in \mathbb{N}$. *For* $\overline{k} = 1, \ldots, N \in \mathbb{N}$, *we denote by*

$$\psi_{1,\overline{k},n} := \Delta(q)\left\{\frac{1}{n^{\overline{k}\alpha}} + (b-a)^{\overline{k}} Te^{-2\lambda n^{(1-\alpha)}}\right\},$$

$0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. *Then*

$$D^*\left(H_n^{\mathcal{F}}(f), f\right) \leq \sum_{\overline{k}=1}^{N} \frac{D^*\left(f^{(\overline{k})}, \widetilde{o}\right)}{\overline{k}!}\psi_{1,\overline{k},n} +$$

$$\omega_1^{(\mathcal{F})}\left(f^{(N)}, \psi_{1,N,n}^{\frac{1}{N}}\right)\psi_{1,N,n}^{\frac{N-1}{N}}\left(\frac{(b-a)}{(N+1)!} + \frac{\psi_{1,N,n}^{\frac{1}{N}}}{2N!} + \frac{\psi_{1,N,n}^{\frac{2}{N}}}{8(b-a)(N-1)!}\right). \tag{136}$$

Clearly it holds $\lim_{n\to\infty} H_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, *uniformly, and* $\lim_{n\to\infty}\left(H_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, *pointwise,* $\forall\ x \in [a,b]$.

Proof. Since $f \in C_{\mathcal{F}}^N([a,b])$, $N \in \mathbb{N}$, we have that $f_{\pm}^{(r)} \in C^N([a,b])$, $\forall\ r \in [0,1]$.

Using (47) we get

$$\left|H_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq \sum_{\overline{k}=1}^{N} \frac{\left\|\left(f_{\pm}^{(r)}\right)^{(\overline{k})}\right\|_{\infty}}{\overline{k}!}\psi_{1,\overline{k},n} +$$

$$\omega_1\left(\left(f_{\pm}^{(r)}\right)^{(N)}, \psi_{1,N,n}^{\frac{1}{N}}\right)\psi_{1,N,n}^{\frac{N-1}{N}}\left(\frac{(b-a)}{(N+1)!} + \frac{\psi_{1,N,n}^{\frac{1}{N}}}{2N!} + \frac{\psi_{1,N,n}^{\frac{2}{N}}}{8(b-a)(N-1)!}\right) \overset{(107)}{=} \tag{137}$$

$$\sum_{\overline{k}=1}^{N} \frac{\left\| \left(f^{(\overline{k})} \right)_{\pm}^{(r)} \right\|_{\infty}}{\overline{k}!} \psi_{1,\overline{k},n} +$$

$$\omega_1 \left(\left(f^{(N)} \right)_{\pm}^{(r)}, \psi_{1,N,n}^{\frac{1}{N}} \right) \psi_{1,N,n}^{\frac{N-1}{N}} \left(\frac{(b-a)}{(N+1)!} + \frac{\psi_{1,N,n}^{\frac{1}{N}}}{2N!} + \frac{\psi_{1,N,n}^{\frac{2}{N}}}{8(b-a)(N-1)!} \right)$$

$$\text{(by (114) and (104))} \leq$$

$$\sum_{\overline{k}=1}^{N} \frac{D^* \left(f^{(\overline{k})}, \widetilde{o} \right)}{\overline{k}!} \psi_{1,\overline{k},n} +$$

$$\omega_1^{(\mathcal{F})} \left(f^{(N)}, \psi_{1,N,n}^{\frac{1}{N}} \right) \psi_{1,N,n}^{\frac{N-1}{N}} \left(\frac{(b-a)}{(N+1)!} + \frac{\psi_{1,N,n}^{\frac{1}{N}}}{2N!} + \frac{\psi_{1,N,n}^{\frac{2}{N}}}{8(b-a)(N-1)!} \right), \tag{138}$$

$\forall\, x \in [a,b]$, $n \in \mathbb{N}$.

Next, we take into account (123).

The theorem is proved. □

Corollary 5.6. Here $f \in C_{\mathcal{F}}^1([a,b])$. Then

$$D^* \left(H_n^{\mathcal{F}}(f), f \right) \leq D^* \left(f^{(1)}, \widetilde{o} \right) \psi_{1,1,n} +$$

$$\frac{1}{2} \omega_1^{(\mathcal{F})} \left(f^{(1)}, \psi_{1,1,n} \right) \left((b-a) + \psi_{1,1,n} + \frac{\psi_{1,1,n}^2}{4(b-a)} \right). \tag{139}$$

Proof. By Theorem 5.5, $N = 1$ case. □

Corollary 5.7. Here $f \in C_{\mathcal{F}}^2([a,b])$. Then

$$D^* \left(H_n^{\mathcal{F}}(f), f \right) \leq D^* \left(f^{(1)}, \widetilde{o} \right) \psi_{1,1,n} + \frac{D^* \left(f^{(2)}, \widetilde{o} \right)}{2} \psi_{1,2,n} +$$

$$\frac{1}{2} \omega_1^{(\mathcal{F})} \left(f^{(2)}, \psi_{1,2,n}^{\frac{1}{2}} \right) \psi_{1,2,n}^{\frac{1}{2}} \left(\frac{(b-a)}{3} + \frac{\psi_{1,2,n}^{\frac{1}{2}}}{2} + \frac{\psi_{1,2,n}}{4(b-a)} \right), \tag{140}$$

$\forall\, n \in \mathbb{N}$.

Proof. By Theorem 5.5, $N = 2$ case. □

Corollary 5.8. Let $f \in C_{\mathcal{F}}^1([a,b])$, $r > 0$. Then

$$D^* \left(H_n^{\mathcal{F}}(f), f \right) \leq D^* \left(f^{(1)}, \widetilde{o} \right) \psi_{1,1,n} +$$

$$\begin{cases} \frac{1}{8r} (2+r)^2 \omega_1^{(\mathcal{F})} \left(f^{(1)}, r\psi_{1,2,n}^{\frac{1}{2}} \right) \psi_{1,2,n}^{\frac{1}{2}}, & \text{if } r \leq 2; \\ \omega_1^{(\mathcal{F})} \left(f^{(1)}, r\psi_{1,2,n}^{\frac{1}{2}} \right) \psi_{1,2,n}^{\frac{1}{2}}, & \text{if } r > 2. \end{cases} \tag{141}$$

Proof. Similar proof as in Theorem 5.5 based on Corollary 2.13. □

We continue with:

Theorem 5.9. *Here $f \in C_{\mathcal{F}}^{N}([a,b])$, $N \in \mathbb{N}$. For $\overline{k} = 1,...,N \in \mathbb{N}$, we denote by*

$$\psi_{2,\overline{k},n} := K(q)\left\{\frac{1}{n^{\overline{k}\alpha}} + (b-a)^{\overline{k}}\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}\right\},$$

$0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. Then

$$D^{*}\left(S_{n}^{s\mathcal{F}}(f), f\right) \leq \sum_{\overline{k}=1}^{N} \frac{D^{*}\left(f^{(\overline{k})}, \widetilde{o}\right)}{\overline{k}!}\psi_{2,\overline{k},n}+$$

$$\omega_{1}^{(\mathcal{F})}\left(f^{(N)}, \psi_{2,N,n}^{\frac{1}{N}}\right)\psi_{2,N,n}^{\frac{N-1}{N}}\left(\frac{(b-a)}{(N+1)!} + \frac{\psi_{2,N,n}^{\frac{1}{N}}}{2N!} + \frac{\psi_{2,N,n}^{\frac{2}{N}}}{8(b-a)(N-1)!}\right). \tag{142}$$

Clearly it holds $\lim_{n\to\infty} S_{n}^{s\mathcal{F}}(f) \overset{D^{*}}{\to} f$, uniformly, and $\lim_{n\to\infty} S_{n}^{s\mathcal{F}}(f)(x) \overset{D}{\to} f(x)$, pointwise, $\forall\, x \in [a,b]$.

Proof. Since $f \in C_{\mathcal{F}}^{N}([a,b])$, $N \in \mathbb{N}$, we have that $f_{\pm}^{(r)} \in C^{N}([a,b])$, $\forall\, r \in [0,1]$.

Using (99) we get

$$\left|S_{n}^{s}\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq \sum_{\overline{k}=1}^{N} \frac{\left\|\left(f_{\pm}^{(r)}\right)^{(\overline{k})}\right\|_{\infty}}{\overline{k}!}\psi_{2,\overline{k},n}+$$

$$\omega_{1}\left(\left(f_{\pm}^{(r)}\right)^{(N)}, \psi_{2,N,n}^{\frac{1}{N}}\right)\psi_{2,N,n}^{\frac{N-1}{N}}\left(\frac{(b-a)}{(N+1)!} + \frac{\psi_{2,N,n}^{\frac{1}{N}}}{2N!} + \frac{\psi_{2,N,n}^{\frac{2}{N}}}{8(b-a)(N-1)!}\right) \overset{(107)}{=} \tag{143}$$

$$\sum_{\overline{k}=1}^{N} \frac{\left\|\left(f^{(\overline{k})}\right)_{\pm}^{(r)}\right\|_{\infty}}{\overline{k}!}\psi_{2,\overline{k},n}+$$

$$\omega_{1}\left(\left(f^{(N)}\right)_{\pm}^{(r)}, \psi_{2,N,n}^{\frac{1}{N}}\right)\psi_{2,N,n}^{\frac{N-1}{N}}\left(\frac{(b-a)}{(N+1)!} + \frac{\psi_{2,N,n}^{\frac{1}{N}}}{2N!} + \frac{\psi_{2,N,n}^{\frac{2}{N}}}{8(b-a)(N-1)!}\right)$$

$$\text{(by (114) and (104))} \;\leq$$

$$\sum_{\overline{k}=1}^{N} \frac{D^{*}\left(f^{(\overline{k})}, \widetilde{o}\right)}{\overline{k}!}\psi_{2,\overline{k},n}+$$

$$\omega_{1}^{(\mathcal{F})}\left(f^{(N)}, \psi_{2,N,n}^{\frac{1}{N}}\right)\psi_{2,N,n}^{\frac{N-1}{N}}\left(\frac{(b-a)}{(N+1)!} + \frac{\psi_{2,N,n}^{\frac{1}{N}}}{2N!} + \frac{\psi_{2,N,n}^{\frac{2}{N}}}{8(b-a)(N-1)!}\right), \tag{144}$$

$\forall\, x \in [a,b]$, $n \in \mathbb{N}$.

Next, we take into account (123).

The theorem is proved. □

Corollary 5.10. Here $f \in C^1_{\mathcal{F}}([a,b])$. Then

$$D^*\left(S_n^{s\mathcal{F}}(f), f\right) \leq D^*\left(f^{(1)}, \widetilde{o}\right)\psi_{2,1,n}+$$

$$\frac{1}{2}\omega_1^{(\mathcal{F})}\left(f^{(1)}, \psi_{2,1,n}\right)\left((b-a)+\psi_{2,1,n}+\frac{\psi_{2,1,n}^2}{4(b-a)}\right). \tag{145}$$

Proof. By Theorem 5.9, $N = 1$ case. □

Corollary 5.11. Here $f \in C^2_{\mathcal{F}}([a,b])$. Then

$$D^*\left(S_n^{s\mathcal{F}}(f), f\right) \leq D^*\left(f^{(1)}, \widetilde{o}\right)\psi_{2,1,n}+\frac{D^*\left(f^{(2)}, \widetilde{o}\right)}{2}\psi_{2,2,n}+$$

$$\frac{1}{2}\omega_1^{(\mathcal{F})}\left(f^{(2)}, \psi_{2,2,n}^{\frac{1}{2}}\right)\psi_{2,2,n}^{\frac{1}{2}}\left(\frac{(b-a)}{3}+\frac{\psi_{2,2,n}^{\frac{1}{2}}}{2}+\frac{\psi_{2,2,n}}{4(b-a)}\right), \tag{146}$$

$\forall\, n \in \mathbb{N}$.

Proof. By Theorem 5.9, $N = 2$ case. □

Corollary 5.12. Let $f \in C^1_{\mathcal{F}}([a,b])$, $r > 0$. Then

$$D^*\left(S_n^{s\mathcal{F}}(f), f\right) \leq D^*\left(f^{(1)}, \widetilde{o}\right)\psi_{2,1,n}+$$

$$\begin{cases} \frac{1}{8r}(2+r)^2\,\omega_1^{(\mathcal{F})}\left(f^{(1)}, r\psi_{2,2,n}^{\frac{1}{2}}\right)\psi_{2,2,n}^{\frac{1}{2}}, & \text{if } r \leq 2; \\ \omega_1^{(\mathcal{F})}\left(f^{(1)}, r\psi_{2,2,n}^{\frac{1}{2}}\right)\psi_{2,2,n}^{\frac{1}{2}}, & \text{if } r > 2. \end{cases} \tag{147}$$

Proof. Similar proof as in Theorem 5.9 based on Corollary 3.13. □

REFERENCES

[1] G.A. Anastassiou, *Rate of convergence of some neural network operators to the unit-univariate case*, Journal of Mathematical Analysis and Application, Vol. 212 (1997), 237-262.

[2] G.A. Anastassiou, *Quantitative Approximation*, Chapmann and Hall/CRC, Boca Raton, New York, 2001.

[3] G.A. Anastassiou, *Higher order Fuzzy Korovkin Theory via inequalities*, Communications in Applied Analysis, 10(2006), No. 2, 359-392.

[4] G.A. Anastassiou, *Fuzzy Korovkin Theorems and Inequalities*, Journal of Fuzzy Mathematics, 15(2007), No. 1, 169-205.

[5] G.A. Anastassiou, *Fuzzy Mathematics: Approximation Theory*, Springer, Heildelberg, New York, 2010.

[6] G.A. Anastassiou, *Intelligent Systems: Approximation by Artificial Neural Networks*, Intelligent Systems Reference Library, Vol. 19, Springer, Heidelberg, 2011.

[7] G.A. Anastassiou, *Intelligent Systems II: Complete Approximation by Neural Network Operators*, Springer, Heidelberg, New York, 2016.

[8] G.A. Anastassiou, *Parametrized, Deformed and General Neural Networks*, Springer, Heidelberg, New York, 2023.

[9] G.A. Anastassiou, *Trigonometric and Hyperbolic Generated Approximation Theory*, World Scientific, Singapore, New York, 2025.

[10] G.A. Anastassiou, *Neural Networks as Positive Linear Operators*, Mathematics, 2025, 13, 1112; https://doi.org/10.3390/math13071112.

[11] G.A. Anastassiou, *Generalized Logistic Neural Networks in Positive Linear Framework*, Symmetry, 2025, 17, 746, htps://doi.org/10.3390/sym17050746.

[12] P. Cardaliaguet, G. Euvrard, *Approximation of a function and its derivarive with a neural network*, Neural Networks 5 (1992), 207-220.

[13] Z. Chen and F. Cao, *The approximation operators with sigmoidal functions*, Computers and Mathematics with Applications, 58 (2009), 758-765.

[14] R. Goetschel Jr., W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets and Systems, 18(1986), 31-43.

[15] S. Haykin, *Neural Networks: A Comprehensive Foundation* (2 ed.), Prentice Hall, New York, 1998.

[16] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems, 24(1987), 301-317.

[17] W. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bulletin of Mathematical Biophysics, 7 (1943), 115-133.

[18] T.M. Mitchell, *Machine Learning*, WCB-McGraw-Hill, New York, 1997.

[19] Wu Congxin, Gong Zengtai, *On Henstock integrals of interval-valued functions and fuzzy valued functions*, Fuzzy Sets and Systems, Vol. 115, No. 3, 2000, 377-391.

[20] C. Wu, Z. Gong, *On Henstock integral of fuzzy-number-valued functions (I)*, Fuzzy Sets and Systems, 120, No. 3, (2001), 523-532.

[21] C. Wu, M. Ma, *On embedding problem of fuzzy number spaces: Part 1*, Fuzzy Sets and Systems, 44 (1991), 33-38.