NUMERICAL SOLUTIONS TO THE SYSTEM OF LINEAR VOLTERRA INTEGRAL EQUATION WITH MIXED TYPE KERNELS

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ABSTRACT. In this study, we propose the Jacobi Spectral Galerkin(JSG) and iterated Jacobi Spectral Galerkin Methods for a system of Linear Volterra Integral Equations(VIE) for mixed type kernels (weakly singular(WS) and smooth). We propose numerical approximation method for solving the system of Volterra integral equation. We improve the order of convergence in iterated Jacobi Spectral Galerkin method over Jacobi spectral Galerkin method. We will also give numerical examples for verifying the theoretical outcomes.

1. INTRODUCTION

The linear VIE of second type is of the form

$$(1.1) \qquad y(t) - \int_0^t k(t,v)y(v)dv = g(t), \quad 0 \le t \le 1,$$

where the source function g and the kernel $k(\cdot,\cdot)$ are provided smooth functions, y is the unidentified function that has to be identified.

Let $\mathbb{X}$ be a Banach Space. Now we define

$$(1.2) \qquad Ky(t) = \int_0^t k(t,v)y(v)dv, \quad t \in [0,1].$$

The equation can be expressed as

$$(1.3) \qquad (I - K)y = g,$$

where $I : \mathbb{X} \to \mathbb{X}$ is an identity operator.

The linear WS Volterra Integral Equation is of the form

$$(1.4) \qquad y - Ky = g,$$

where

$$(1.5) \qquad Ky(t) = \int_0^t k(t,v)y(v)dv, \ 0 \le v < t \le 1,$$

with the kernel

$$(1.6) \qquad k(t,v) = \begin{cases} m(t,v)(t-v)^{-\gamma}, & \text{if } 0 < \gamma < 1 \\ m(t,v)\log(t-v), & \text{if } \gamma = 1. \end{cases}$$

The system of Volterra-Linear integral equation of the second kind with smooth or WS kernel is given as follows:

(1.7)

$$y_i(t) = g_i(t) + \int_0^t W_{i1}(t,\nu)y_1(\nu)d\nu + \int_0^t W_{i2}(t,\nu)y_2(\nu)d\nu + \ldots + \int_0^t W_{im}(t,\nu)y_m(\nu)d\nu \ 1 \le i \le m,$$

or

$$(1.8) \qquad y_i(t) = g_i(t) + \sum_{j=1}^m \int_0^t W_{ij}(t,\nu)y_j(\nu)d\nu \ 1 \le i,j \le m \ t \in [0,1],$$

with

$$W_{ij} = \left\{ \begin{array}{ll} d_{ij} \text{ is smooth kernel,} & \text{for } 1 \le j \le l \\ (t-\nu)^{-\gamma}d_{ij} \text{ is WS kernel,} & \text{for } l \le j \le m. \end{array} \right\}$$

Here d_{ij} and $g_i's$ are smooth functions and $0 < \gamma < 1$.

where $g_i : [0,1] \rightarrow [0,1]$ is an inhomogeneous function in the Banach space $\mathbb{X} = L^\infty[0,1]$, which is sufficiently smooth, the unknown function $y_i : [0,1] \rightarrow [0,1]$ in $\mathbb{X}$, is to be determined. In the Eq. (1.7) kernel W_{ij} is WS kernel for $l \le j \le m$ because $(t-\nu)^{-\gamma}$ makes our kernel unbounded at some $\nu = t$. In the sciences and engineering, systems of linear VIEs and their solutions play a crucial role. A system of VIEs may be utilised to simulate most physical problems, including biomedical applications in population trends and engineering models. Numerous applications of the second class of VIEs are found in [6], [7], [8], including the Dirichlet problem, reactor theory, electrodynamics, astronomy, and problems with heat transfer. For the linear VIE of the second kind with a smooth kernel, Moumita et.al.[1] has addressed the Legendre polynomial-based projection techniques for applying the orthogonal and interpolatory projection operators. In order to solve WS linear VIE in the piecewise polynomial subspaces based on the graded mesh, Kant et.al.[2] has addressed Galerkin, multi-Galerkin, and their iterated versions and found the convergence analysis. We see that the exact solution y of these type of WS VIE is either sufficiently smooth or exhibits the same behaviour as y(t) , which has a singularity at t = 0. In Kant et.al. [3] addressed the JSG method and its iterated version for linear WS volterra integral equation with algebraic type kernel and he considered the two cases of the solution when it is smooth or non-smooth and obtained the superconvergence analysis. In order to approximate the VIE for smooth and WS kernels to find the superconvergence

analysis, Samiran chakraborty et.al [4] has studied the JSG method and its iterated variations. In this article, we study the JSG method and its iterated version to approximating the system of linear VIE with mixed kind of kernels (weakly as well as smooth) and obtain the convergence analysis. For iterated Galerkin appproximation we get the order $\mathcal{O}\left(N^{-2q}\right)$ when exact solution is smooth in both norm L^2 and ∞ norm here q denotes the smoothness of the exact solution. We will also illustrate some numerical examples to verify the theoretical outcomes.

2. JSG METHOD FOR SYSTEM OF VOLTERRA INTEGRAL EQUATION

In the section we will cover the Galerkin and iterated Galerkin methods for the system of Linear Volterra integral equation with mixed type kernels. Consider the system of second kind Volterra integral equation with a mixed kind kernel in a Banach space $\mathbb{X} = L^\infty[0, 1]$

$$(2.1) \qquad y_i(t) = g_i(t) + \sum_{j=1}^{m} \int_0^t W_{ij}(t, \nu)y_j(\nu)d\nu \ 1 \leq i \leq m \ t \in [0, 1], \ \gamma \in (0, 1).$$

Now we define the integral operator

$$(2.2) \qquad K_{ij}y_j(t) = \int_0^t W_{ij}(t, \nu)y_j(\nu)d\nu \ 1 \leq i, j \leq m \ t \in [0, 1].$$

Now we use the following transformation :

$$t = \frac{1}{2}T(1 + x), \quad x = \frac{2t}{T} - 1,$$
$$\nu = \frac{1}{2}T(1 + s), \quad s = \frac{2\nu}{T} - 1,$$

and denote

$$(2.3) \qquad u_i(x) = y_i\left(\frac{1}{2}T(1 + x)\right), \quad f_i(x) = g_i\left(\frac{1}{2}T(1 + x)\right).$$

Now for $1 \leq j \leq l$, we denote the smooth kernel

$$(2.4) \qquad k_{ij}(x, s) = \left(\frac{T}{2}\right) d_{ij}\left(\frac{1}{2}T(1 + x), \frac{1}{2}T(1 + s)\right),$$

and for $l \leq j \leq m$, the WS kernel

$$(2.5) \qquad k_{ij}(x, s) = \left(\frac{T}{2}\right)^{1-\gamma} d_{ij}\left(\frac{1}{2}T(1 + x), \frac{1}{2}T(1 + s)\right).$$

So after using the above transformation integral equation becomes

(2.6)

$$u_i(x) = f_i(x) + \sum_{j=1}^{l} \int_{-1}^{x} k_{ij}(x, s)u_j(s)ds + \sum_{j=l}^{m} \int_{-1}^{x} (x - s)^{-\gamma} k_{ij}(x, s)u_j(s)ds \ 1 \leq i \leq m.$$

Now to find the better convergence rates, we change the interval of integration from $[-1, x]$ to $[-1, 1]$ by the following transformation $s(.,.) : ([-1, 1] \times [-1, 1]) \to [-1, 1]$.

$$(2.7) \qquad s(x, \theta) = \frac{1+x}{2}\theta + \frac{x-1}{2}.$$

So we get

(2.8)
$$u_i(x) = f_i(x) + \sum_{j=1}^{l} \int_{-1}^{1} \tilde{k}_{ij}(x, s(x, \theta))u_j(s(x, \theta))d\theta + \sum_{j=l}^{m} \int_{-1}^{1} (1-\theta)^{-\gamma}\tilde{k}_{ij}(x, s(x, \theta))u_j(s(x, \theta))d\theta, \ 1 \le i \le m.$$

For $1 \le j \le l$, we denote the integral operator

$$(2.9) \qquad K_{ij}u_j(x) = \int_{-1}^{1} \tilde{k}_{ij}(x, s(x, \theta))u_j(s(x, \theta))d\theta, \ x \in [-1, 1].$$

Here, $\tilde{k}_{ij}(x, s(x, \theta)) = \left(\frac{1+x}{2}\right) k_{ij}(x, s(x, \theta))$ is smooth function.
Also for $l \le j \le m$, we denote the integral operator

$$(2.10) \qquad K_{ij}u_j(x) = \int_{-1}^{1} (1-\theta)^{-\gamma}\tilde{k}_{ij}(x, s(x, \theta))u_j(s(x, \theta))d\theta, \ x \in [-1, 1].$$

Where, $\tilde{k}_{ij}(x, s(x, \theta)) = \left(\frac{1+x}{2}\right)^{1-\gamma} k_{ij}(x, s(x, \theta))$ is sufficiently differential function with respect to θ. So the system of equation can be written as

$$(2.11) \qquad u_i(x) = \sum_{j=1}^{m} K_{ij}u_j(x) + f_i(x), \ 1 \le i \le m.$$

The system of equation in matrix form can be written as

$$(2.12) \qquad \mathbf{I}_m\mathbf{U} - \mathbf{K}\mathbf{U} = \mathbf{F}.$$

Here

$$(2.13) \quad \mathbf{I}_m = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \cdots & & \cdots & \\ 0 & 0 & \cdots & I \end{bmatrix}_{m \times m}, \quad \mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1m} \\ K_{21} & K_{22} & \cdots & K_{2m} \\ \cdots & & \cdots & \\ K_{m1} & K_{m2} & \cdots & K_{mm} \end{bmatrix}_{m \times m},$$

and

$$(2.14) \qquad \mathbf{U} = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}^T_{m \times 1}, \quad \mathbf{F} = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \end{bmatrix}^T_{m \times 1}.$$

Let $\mathbb{X}_N = \text{span}\{e_0, e_1, e_2, e_3, \ldots, e_N\}$ be the subspace of Jacobi polynomials of degree $\le N$ on $[-1, 1]$, here $e_j(x)$ is the j-th Jacobi polynomial corresponding to the weight function

$$(2.15) \qquad \omega^{\alpha,\beta}(x) = (1+x)^{\beta}(1-x)^{\alpha}, \ \beta, \alpha > -1.$$

Throughout this paper we will take $\alpha = -\gamma$ and $\beta = 0$ because in the integral operator (2.10) kernel has singularity at 1, so to incorporate the singularity in the

weight function.

For any $v \in \mathcal{C}^m([0,1])$, we write

$$(2.16) \qquad \|y\|_{m,\infty} = \max\left\{\left\|y^{(j)}\right\|_\infty : 0 \leq j \leq m\right\},$$

where $y^{(j)}$ denotes the jth derivative of y.

Let us give the basic definitions which will be used in our main results:

Definition 2.1. (Norm convergence): If $\mathbb{X}$ is a Banach space, the set $\mathcal{BL}(\mathbb{X})$ is the space of all bounded linear operators on $\mathbb{X}$ and $\mathcal{F}_n, \mathcal{F} \in \mathcal{BL}(\mathbb{X})$, then $\mathcal{F}_n$ is said to be norm convergent to $\mathcal{F}$ if $\|\mathcal{F}_n - \mathcal{F}\| \to 0$, as $n \to \infty$. It is denoted by $\mathcal{F}_n \xrightarrow{n} \mathcal{F}$.

Definition 2.2. (Orthogonal projection): If $\omega^{\alpha,\beta}$, $\beta > -1, \alpha > -1$ is a weight function as given in (2.15). Then the orthogonal projection $\mathcal{P}_N^{\alpha,\beta} : \mathbb{X} \to \mathbb{X}_N$, is given by

$$(2.17) \qquad \left\langle \mathcal{P}_N^{\alpha,\beta} y, y_N \right\rangle_{\omega^{\alpha,\beta}} = \langle y, y_N \rangle_{\omega^{\alpha,\beta}}, \quad \forall y \in L^2_{\omega^{\alpha,\beta}}, y_N \in \mathbb{X}_N,$$

here

$$\langle y_1, y_2 \rangle_{\omega^{\alpha,\beta}} = \int_{-1}^1 y_1(t) y_2(t) \omega^{\alpha,\beta}(t) dt.$$

$$L^2_{\omega^{\alpha,\beta}} = \left\{y : y \text{ is measurable and } \|y\|_{\omega^{\alpha,\beta}} < \infty\right\},$$

$$(2.18) \qquad \|y\|_{\omega^{\alpha,\beta}} = \left(\int_{-1}^1 y^2(t) \omega^{\alpha,\beta}(t) dt\right)^{\frac{1}{2}}.$$

So, we define

$$H^m_{\omega^{\alpha,\beta}}(\Lambda) = \left\{y : D^k y \in L^2_{\omega^{\alpha,\beta}}(\Lambda), 0 \leq k \leq m\right\},$$

and the seminorm on $H^m_{\omega^{\alpha,\beta}}(\Lambda)$ is given by

$$(2.19) \qquad \|y\|_{H^{m,N}_{\omega^{\alpha\beta}}} = \left(\sum_{k=\min(m,N+1)}^m \left\|\frac{d^k y}{dx^k}\right\|^2_{\omega^{a,\beta}}\right)^{\frac{1}{2}}.$$

According to [9], [10], the following lemma gives the essential properties of the orthogonal projection operator.

Lemma 2.3. *If $\mathcal{P}_N^{\alpha,\beta} : \mathbb{X} \to \mathbb{X}_N$ denote the orthogonal projection defined by (2.17). Then the projection $\mathcal{P}_N^{\alpha,\beta}$ satisfies the following properties. So for any $y \in \mathcal{C}[-1,1]$, we get*

$(i)\ \left\|\mathcal{P}_N^{\alpha,\beta} y\right\|_{\omega^{\alpha,\beta}} \leq p\|y\|_\infty$ $(ii)\ \left\|\mathcal{P}_N^{\alpha,\beta} y\right\|_{\omega^{\alpha,\beta}} \leq \|y\|_{\omega^{\alpha,\beta}}$ $(iii)\ \left\|\mathcal{P}_N^{\alpha,\beta}\right\|_\infty \leq c(\log N).$

Lemma 2.4. *If $\mathcal{P}_N^{\alpha,\beta}$ is the orthogonal projection defined by (2.17). If $y \in \mathcal{C}^m[-1,1]$ and $m \geq 1$, then we get*

$$(2.20) \qquad (i) \;\; \|y - \mathcal{P}_N^{\alpha,\beta} y\|_{\omega^{\alpha,\beta}} \leq CN^{-m} |y|_{H_{\omega^{\alpha,\beta}}^{m;N}},$$

$$(2.21) \qquad (ii) \;\; \|y - \mathcal{P}_N^{\alpha,\beta} y\|_\infty \leq CN^{\frac{3}{4}-m} |y|_{H_{\omega^{\alpha,\beta}}^{m;N}}.$$

Note that C is a constant independent of N.

3. NUMERICAL ALGORITHM FOR SOLVING THE SYSTEM OF VIE FOR MIXED TYPE KERNEL

The Galerkin method to solve is seeking $u_1^n, u_2^n, \ldots, u_m^n \in \mathbb{X}_n$ such that

$$(3.1) \qquad u_i^n - \sum_{j=1}^m \mathcal{P}_N^{\alpha,\beta} K_{ij} u_j^n = \mathcal{P}_N^{\alpha,\beta} f_i, \quad 1 \leq i \leq m.$$

Then (3.1) can be written as

$$(3.2) \qquad \mathbf{I}_m \mathbf{U}_n - \mathbf{K}_n \mathbf{U}_n = \mathbf{F}_n,$$

where

$$(3.3)$$

$$\mathbf{K}_n = \begin{bmatrix} \mathcal{P}_N^{\alpha,\beta} K_{11} & \mathcal{P}_N^{\alpha,\beta} K_{12} & \ldots & \mathcal{P}_N^{\alpha,\beta} K_{1m} \\ \mathcal{P}_N^{\alpha,\beta} K_{21} & \mathcal{P}_N^{\alpha,\beta} K_{22} & \ldots & \mathcal{P}_N^{\alpha,\beta} K_{2m} \\ \ldots & \ldots & & \\ \mathcal{P}_N^{\alpha,\beta} K_{m1} & \mathcal{P}_N^{\alpha,\beta} K_{m2} & \ldots & \mathcal{P}_N^{\alpha,\beta} K_{mm} \end{bmatrix}_{m \times m} , \quad \mathbf{U}_n = \begin{bmatrix} u_1^n \\ u_2^n \\ \ldots \\ u_m^n \end{bmatrix}_{m \times 1} .$$

Now the iterated Galerkin approximation is defined as

$$(3.4) \qquad \tilde{u}_i^n = \sum_{j=1}^m \mathcal{K}_{ij} u_j^n + f_i, \quad 1 \leq i \leq m.$$

So we can see that $u_i^n = P_n \tilde{u}_i^n, i = 1, 2, \ldots, m$. And we can write the above system as

$$(3.5) \qquad \tilde{u}_i^n - \sum_{j=1}^m K_{ij} \mathcal{P}_N^{\alpha,\beta} \tilde{u}_j^n = f_i, \quad 1 \leq i \leq m,$$

which we write as

$$(3.6) \qquad \mathbf{I}_m \tilde{\mathbf{U}}_n - \tilde{\mathbf{K}}_n \tilde{\mathbf{U}}_n = \mathbf{F},$$

(3.7)

$$\tilde{\mathbf{K}}_n = \begin{bmatrix} K_{11}\mathcal{P}_N^{\alpha,\beta} & K_{12}\mathcal{P}_N^{\alpha,\beta} & \cdots & K_{1m}\mathcal{P}_N^{\alpha,\beta} \\ K_{21}\mathcal{P}_N^{\alpha,\beta} & K_{22}\mathcal{P}_N^{\alpha,\beta} & \cdots & K_{2m}\mathcal{P}_N^{\alpha,\beta} \\ \cdots & & \cdots & \\ K_{m1}\mathcal{P}_N^{\alpha,\beta} & K_{m2}\mathcal{P}_N^{\alpha,\beta} & \cdots & K_{mm}\mathcal{P}_N^{\alpha,\beta} \end{bmatrix}_{m\times m} \quad \text{and} \quad \tilde{\mathbf{U}}_n = \begin{bmatrix} \tilde{u}_1^n \\ \tilde{u}_2^n \\ \cdots \\ \tilde{u}_m^n \end{bmatrix}_{m\times 1}.$$

Now we take the system of VIE

$$u_i(x) = \sum_{j=1}^{m} \int_0^1 k_{ij}(x,t)u_j(t)dt + f_i(x) \quad i = 1, 2, \ldots m$$

$$u_1(x) = \int_0^1 k_{11}(x,t)u_1(t)dt + \int_0^1 k_{12}(x,t)u_2(t)dt + \ldots + \int_0^1 k_{1m}(x,t)u_m(t)dt + f_1(x)$$

$$\vdots \qquad \vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \vdots$$

$$u_m(x) = \int_0^1 k_{m1}(x,t)u_1(t)dt + \int_0^1 k_{m2}(x,t)v_2(t)dt + \ldots + \int_0^1 k_{mm}(x,t)u_m(t) + f_m(x),$$

(3.8)
$$\text{so } u_i(x) = \sum_{j=1}^{m} K_{ij}(x,t)u_j(x) + f_i(x).$$

Now, applying the projection operator we get

(3.9)
$$\mathcal{P}_N^{\alpha,\beta}u_i(x) = \sum_{j=1}^{m} \mathcal{P}_N^{\alpha,\beta}K_{ij}(x,t)u_j(t) + \mathcal{P}_N^{\alpha,\beta}f_i$$

$$u_i^n(x) = \sum_{j=1}^{m} \int_0^1 \mathcal{P}_N^{\alpha,\beta}k_{ij}(x,t)u_j(t)dt + \mathcal{P}_N^{\alpha,\beta}f_i$$

(3.10)
$$\text{as } u_i^n(x) \in \text{span}\,\{e_1, e_2, \ldots e_n\} \implies u_i^n(x) = \sum_{k=1}^{n} \alpha_k^i e_k.$$

Now, using the equations (3.9) and (3.10) we get

(3.11)
$$\sum_{k=1}^{n} \alpha_k^i e_k - \sum_{j=1}^{m} \mathcal{P}_N^{\alpha,\beta}K_{ij}(x,t)(\sum_{k=1}^{n} \alpha_k^j e_k) = \mathcal{P}_N^{\alpha,\beta}f_i \quad \forall i = 1, 2, \cdots, m$$

$$\sum_{k=1}^{n} \alpha_k^i e_k - \mathcal{P}_N^{\alpha,\beta}K_{i1}(\sum_{k=1}^{n} \alpha_k^1 e_k) - \mathcal{P}_N^{\alpha,\beta}K_{i2}(\sum_{k=1}^{n} \alpha_k^2 e_k) - \cdots - \mathcal{P}_N^{\alpha,\beta}K_{im}(\sum_{k=1}^{n} \alpha_k^m e_k) = \mathcal{P}_N^{\alpha,\beta}f_i \quad \forall i = 1, 2, \cdots, m.$$

Now we can write the above equation for each i

$$\sum_{k=1}^{n} \alpha_k^1 e_k - \mathcal{P}_N^{\alpha,\beta} K_{11}(\sum_{k=1}^{n} \alpha_k^1 e_k) - \mathcal{P}_N^{\alpha,\beta} K_{12}(\sum_{k=1}^{n} \alpha_k^2 e_k) - \cdots - \mathcal{P}_N^{\alpha,\beta} K_{1m}(\sum_{k=1}^{n} \alpha_k^m e_k) = \mathcal{P}_N^{\alpha,\beta} f_1$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\sum_{k=1}^{n} \alpha_k^m e_k - \mathcal{P}_N^{\alpha,\beta} K_{m1}(\sum_{k=1}^{n} \alpha_k^1 e_k) - \mathcal{P}_N^{\alpha,\beta} K_{m2}(\sum_{k=1}^{n} \alpha_k^2 e_k) - \cdots - \mathcal{P}_N^{\alpha,\beta} K_{mm}(\sum_{k=1}^{n} \alpha_k^m e_k) = \mathcal{P}_N^{\alpha,\beta} f_m.$$

So the above equations can be written in this form

$$(I - \mathcal{P}_N^{\alpha,\beta} K_{11})(\sum_{k=1}^{n} \alpha_k^1 e_k) - \mathcal{P}_N^{\alpha,\beta} K_{12}(\sum_{k=1}^{n} \alpha_k^2 e_k) - \cdots - \mathcal{P}_N^{\alpha,\beta} K_{1m}(\sum_{k=1}^{n} \alpha_k^m e_k) = \mathcal{P}_N^{\alpha,\beta} f_1$$

$$- \mathcal{P}_N^{\alpha,\beta} K_{21}(\sum_{k=1}^{n} \alpha_k^1 e_k) + (I - \mathcal{P}_N^{\alpha,\beta} K_{22})(\sum_{k=1}^{n} \alpha_k^2 e_k) - \cdots - \mathcal{P}_N^{\alpha,\beta} K_{2m}(\sum_{k=1}^{n} \alpha_k^m e_k) = \mathcal{P}_N^{\alpha,\beta} f_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$- \mathcal{P}_N^{\alpha,\beta} K_{m1}(\sum_{k=1}^{n} \alpha_k^1 e_k) - \mathcal{P}_N^{\alpha,\beta} K_{m2}(\sum_{k=1}^{n} \alpha_k^2 e_k) - \cdots + (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})(\sum_{k=1}^{n} \alpha_k^m e_k) = \mathcal{P}_N^{\alpha,\beta} f_m.$$

Now we take the inner product with e_r for each $r = 1, 2, 3...n$, where $e_r \in \{e_1, e_2, \ldots e_n\}$

$$\left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})(\sum_{k=1}^{n} \alpha_k^1 e_k), e_1 \right\rangle - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{12}(\sum_{k=1}^{n} \alpha_k^2 e_k), e_1 \right\rangle - \cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}(\sum_{k=1}^{n} \alpha_k^m e_k), e_1 \right\rangle = \langle f_1, e_1 \rangle$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{21}(\sum_{k=1}^{n} \alpha_k^1 e_k), e_1 \right\rangle + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{22})(\sum_{k=1}^{n} \alpha_k^2 e_k), e_1 \right\rangle - \cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{2m}(\sum_{k=1}^{n} \alpha_k^m e_k), e_1 \right\rangle = \langle f_2, e_1 \rangle$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1}(\sum_{k=1}^{n} \alpha_k^1 e_k), e_1 \right\rangle - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m2}(\sum_{k=1}^{n} \alpha_k^2 e_k), e_1 \right\rangle - \cdots + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})(\sum_{k=1}^{n} \alpha_k^m e_k), e_1 \right\rangle = \langle f_m, e_1 \rangle$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})(\sum_{k=1}^{n} \alpha_k^1 e_k), e_n \right\rangle - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{12}(\sum_{k=1}^{n} \alpha_k^2 e_k), e_n \right\rangle - \cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}(\sum_{k=1}^{n} \alpha_k^m e_k), e_n \right\rangle = \langle f_1, e_n \rangle$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{21}(\sum_{k=1}^{n} \alpha_k^1 e_k), e_n \right\rangle + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{22})(\sum_{k=1}^{n} \alpha_k^2 e_k), e_n \right\rangle - \cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{2m}(\sum_{k=1}^{n} \alpha_k^m e_k), e_n \right\rangle = \langle f_2, e_n \rangle$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1}(\sum_{k=1}^{n} \alpha_k^1 e_k), e_n \right\rangle - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m2}(\sum_{k=1}^{n} \alpha_k^2 e_k), e_n \right\rangle - \cdots + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})(\sum_{k=1}^{n} \alpha_k^m e_k), e_n \right\rangle = \langle f_m, e_n \rangle$$

$$\left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})e_1, e_1 \right\rangle \alpha_1^1 + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})e_2, e_1 \right\rangle \alpha_2^1 + \cdots + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})e_n, e_1 \right\rangle \alpha_n^1$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{12}e_1, e_1 \right\rangle \alpha_1^2 - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{12}e_2, e_1 \right\rangle \alpha_2^2 - \cdots\cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{12}e_n, e_1 \right\rangle \alpha_n^2 - \cdots$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}e_1, e_1 \right\rangle \alpha_1^m - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}e_2, e_1 \right\rangle \alpha_2^m - \cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}e_n, e_1 \right\rangle \alpha_n^m = \langle f_1, e_1 \rangle$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1}e_1, e_1 \right\rangle \alpha_1^1 - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1})e_2, e_1 \right\rangle \alpha_2^1 - \cdots\cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1}e_n, e_1 \right\rangle \alpha_n^1$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m2}e_1, e_1 \right\rangle \alpha_1^2 - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m2}e_2, e_1 \right\rangle \alpha_2^2 - \cdots\cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m2}e_n, e_1 \right\rangle \alpha_n^2 - \cdots$$

$$+ \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})e_1, e_1 \right\rangle \alpha_1^m + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})e_2, e_1 \right\rangle \alpha_2^m + \cdots + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})e_n, e_1 \right\rangle \alpha_n^m = \langle f_m, e_1 \rangle$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})e_1, e_n \right\rangle \alpha_1^1 + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})e_2, e_n \right\rangle \alpha_2^1 + \cdots + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})e_n, e_n \right\rangle \alpha_n^1$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{12}e_1, e_n \right\rangle \alpha_1^2 - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{12}e_2, e_n \right\rangle \alpha_2^2 - \cdots\cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{12}e_n, e_n \right\rangle \alpha_n^2 - \cdots$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}e_1, e_n \right\rangle \alpha_1^m - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}e_2, e_n \right\rangle \alpha_2^m - \cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{12}e_n, e_n \right\rangle \alpha_n^m = \langle f_1, e_n \rangle$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1}e_1, e_n \right\rangle \alpha_1^1 - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1})e_2, e_n \right\rangle \alpha_2^1 - \cdots\cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1}e_n, e_n \right\rangle \alpha_n^1$$

$$- \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m2}e_1, e_n \right\rangle \alpha_1^2 - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m2}e_2, e_n \right\rangle \alpha_2^2 - \cdots\cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m2}e_n, e_n \right\rangle \alpha_n^2 - \cdots$$

$$+ \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})e_1, e_n \right\rangle \alpha_1^m + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})e_2, e_n \right\rangle \alpha_2^m + \cdots + \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})e_n, e_n \right\rangle \alpha_n^m = \langle f_m, e_n \rangle.$$

So we write the above equations in matrix form

$$\mathbf{J} = \begin{bmatrix} \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})e_1, e_1 \right\rangle & \cdots \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})e_n, e_1 \right\rangle \cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}e_1, e_1 \right\rangle \cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}e_n, e_1 \right\rangle \\ \vdots \qquad\qquad\qquad \vdots \\ - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1}e_1, e_1 \right\rangle \cdots \quad - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1}e_n, e_1 \right\rangle \cdots \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})e_1, e_1 \right\rangle \cdots \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})e_n, e_1 \right\rangle \\ \vdots \qquad\qquad\qquad \vdots \\ \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})e_1, e_n \right\rangle & \cdots \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{11})e_n, e_n \right\rangle \cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}e_1, e_n \right\rangle \cdots - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{1m}e_n, e_n \right\rangle \\ \vdots \qquad\qquad\qquad \vdots \\ - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1}e_1, e_n \right\rangle \cdots \quad - \left\langle \mathcal{P}_N^{\alpha,\beta} K_{m1}e_n, e_n \right\rangle \cdots \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})e_1, e_n \right\rangle \cdots \left\langle (I - \mathcal{P}_N^{\alpha,\beta} K_{mm})e_n, e_n \right\rangle \end{bmatrix}_{mn \times mn}$$

$$(3.12) \qquad \mathbf{G} = \begin{bmatrix} \langle f_1, e_1 \rangle \\ \langle f_2, e_1 \rangle \\ \ldots \\ \langle f_m, e_1 \rangle \\ \langle f_1, e_2 \rangle \\ \langle f_2, e_2 \rangle \\ \ldots \\ \langle f_m, e_2 \rangle \\ \ldots \\ \ldots \\ \langle f_1, e_n \rangle \\ \langle f_2, e_n \rangle \\ \ldots \\ \langle f_m, e_n \rangle \end{bmatrix}_{mn \times 1}$$

$$(3.13) \qquad \mathbf{N} = \begin{bmatrix} \alpha_1^1 & \alpha_2^1 & \cdots & \alpha_n^1 & \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \cdots & \cdots & \alpha_1^m & \alpha_2^m & \cdots & \alpha_n^m \end{bmatrix}^T_{mn \times 1}$$

$$(3.14) \qquad \mathbf{J N = G}.$$

So we can find the $\mathbf{N}$. Hence we can find the approximate solution.

Theorem 3.1. *If u_i is the exact solution for $i = 1, 2, \ldots, m$, of the system of the linear integral Eq. 2.8, then we get*

$$(3.15) \qquad \omega_\infty(u_i, \delta) \leq C\delta^{1-\gamma}, \quad i = 1, 2, \ldots, m,$$

here ω_∞ denote the modulus of continuity of u_i.

Proof. The proof is similar to Kant et al. [3].

Lemma 3.2. *If u_i is the solution of the system (2.8), which is non-smooth i.e., $u_i(x) \sim (1 + x)^{1-\gamma}$, for $0 < \gamma < 1$, then we have*

$$\left\| \left(I - \mathcal{P}_N^{-\gamma,0} \right) u_i \right\|_{\omega^{\alpha-1,0}} = \mathcal{O} \left(N^{-(1-\gamma)} \right)$$

$$\left\| \left(I - \mathcal{P}_N^{-\gamma,0} \right) u_i \right\|_\infty = \mathcal{O} \left(N^{-(1-\gamma)} \log N \right)$$

Proof: Follow from [5].

Let 1 be not an eigenvalue of the operators K_{ii}, for $i = 1, 2, \ldots, m$, and

(3.16)
$$\left\| \det\left(\mathbf{I}_m - \mathbf{K}\right) - \prod_{i=1}^{m} \left(I - K_{ii}\right) \right\|_{\infty} < \frac{1}{\left\| \left(I - K_{mm}\right)^{-1} \cdots \left(I - K_{22}\right)^{-1} \left(I - K_{11}\right)^{-1} \right\|_{\infty}}.$$

Then $\left(\mathbf{I}_m - \mathbf{K}\right)^{-1}$ exists and bounded, i.e., there exists some $L > 0$ such that $\left\| \left(\mathbf{I}_m - \mathbf{K}\right)^{-1} \right\|_{\infty} \leq L < \infty$.

we get this from [4].

Theorem 3.3. *If $\tilde{\mathbf{K}}_n$ is the matrice of operators as given above, then for sufficiently large n, $\left(\mathbf{I}_m - \tilde{\mathbf{K}}_n\right)^{-1}$ exist and uniformly bounded in infinity norm, i.e., $\exists$ constants $L_2 > 0$ such that $\left\| \left(\mathbf{I}_m - \tilde{\mathbf{K}}_n\right)^{-1} \right\|_{\infty} \leq L_2 < \infty$.*

Theorem 3.4. *If $\mathbf{U} = \left(u_1, u_2, \ldots, u_m\right)^T$ is the exact solution of the system (2.12) and $\tilde{\mathbf{U}}_n = \left(\tilde{u}_1^n, \tilde{u}_2^n, \ldots, \tilde{u}_m^n\right)^T$ be the iterated Galerkin approximation of $\mathbf{U}$. Then these results holds*
(1) when solution is smooth :

(3.17)
$$\left\| \tilde{\mathbf{U}}_n - \mathbf{U} \right\|_{\infty} = \mathcal{O}\left(N^{-2q}\right),$$

(3.18)
$$\left\| \tilde{\mathbf{U}}_n - \mathbf{U} \right\|_{\omega^{\alpha,\beta}} = \mathcal{O}\left(N^{-2q}\right).$$

(2) when solution is non-smooth :

(3.19)
$$\left\| \tilde{\mathbf{U}}_n - \mathbf{U} \right\|_{\infty} = \mathcal{O}(N^{-(q+1-\gamma)}),$$

(3.20)
$$\left\| \tilde{\mathbf{U}}_n - \mathbf{U} \right\|_{\omega^{\alpha,\beta}} = \mathcal{O}(N^{-(q+1-\gamma)}).$$

Theorem 3.5. *Let $\mathbf{U} = \left(u_1, u_2, \ldots, u_m\right)^T$ is the exact solution of (2.12) and $\mathbf{U}_N = \left(u_1^N, u_2^N, \ldots, u_m^N\right)^T$ be the Galerkin approximation of U. Then there hold:*
(i) If exact solution is smooth then, we have

(3.21)
$$\left\| \mathbf{U}_N - \mathbf{U} \right\|_{\infty} = \mathcal{O}\left(N^{\frac{3}{4}-q}\right),$$

(3.22)
$$\left\| \mathbf{U}_N - \mathbf{U} \right\|_{\omega^{\alpha,\beta}} = \mathcal{O}\left(N^{-q}\right).$$

(ii) If exact solution is non-smooth then, we have

(3.23)
$$\left\| \mathbf{U}_N - \mathbf{U} \right\|_{\infty} = \mathcal{O}\left(N^{-(1-\gamma)}\right),$$

(3.24)
$$\left\| \mathbf{U}_N - \mathbf{U} \right\|_{\omega^{\alpha,\beta}} = \mathcal{O}\left(N^{-(1-\gamma)}\right).$$

We note that Jacobi spectral iterated Galerkin (JSIG) approximate solution convergence rates improve over JSG method in all cases.

Hence from above theorem, convergence rates of the JSIG approximate solutions are higher than the JSG method in all cases.

4. Numerical results

Here, numerical data are provided to better illustrate the theory's results. We choose Jacobi polynomials as the basis functions for the estimate space $\mathbb{X}_N$. The errors that were found during the computation of the estimated solutions $\mathbf{U}_N$ and $\tilde{\mathbf{U}}_N$ of the JSG and the iterated JSG in the infinite and weighted L2 norms, respectively, are mentioned below. Then we designate U as the precise solution

$$(4.1) \qquad \|\mathbf{U}_N - \mathbf{U}\|_\infty = \mathcal{O}\left(N^{-a_i}\right), \ \|\mathbf{U}_N - \mathbf{U}\| = \mathcal{O}\left(N^{-\bar{a}_i}\right)$$

$$(4.2) \qquad \left\|\tilde{\mathbf{U}}_N - \mathbf{U}\right\|_\infty = \mathcal{O}\left(N^{-b_i}\right), \ \left\|\tilde{\mathbf{U}}_N - \mathbf{U}\right\| = \mathcal{O}\left(N^{-\bar{b}_i}\right)$$

Example 4.1. For the smooth kernels we take the system of linear second kind VIE as given below

$$y_1(t) + \int_0^t \left(e^{t-v} y_1(v) + \cos(t-v) y_2(v)\right) dv = \cosh t + t \sin t$$

$$y_2(t) + \int_0^t \left(e^{t+v} y_1(v) + t \cos v y_2(v)\right) dv = 2 \sin t + t \left(\sin^2 t + e^t\right)$$

with exact solutions $u_1(t) = e^{-t}, u_2(t) = 2\sin t$. Using transformations, we get

$$u_1(x) + \int_{-1}^1 \frac{1+x}{4} \left(e^{\left(\frac{1+x}{2}\frac{1-\theta}{2}\right)} u_1(s(x,\theta)) + \cos\left(\frac{1+x}{2}\frac{1-\theta}{2}\right) u_2(s(x,\theta))\right) d\theta = f_1(x)$$

$$u_2(x) + \int_{-1}^1 \frac{1+x}{4} \left(e^{\left(\frac{1+x}{2}\frac{3+\theta}{2}\right)} u_1(s(x,\theta)) + \frac{1+x}{2}\cos\left(\frac{1+x}{2}\frac{1+\theta}{2}\right) u_2(s(x,\theta))\right) d\theta = f_2(x),$$

where $f_1(x) = \cosh\frac{1+x}{2} + \frac{1+x}{2}\sin\frac{1+x}{2}$, and $f_2(x) = 2\sin\frac{1+x}{2} + \frac{1+x}{2}\left(\sin^2\frac{1+x}{2} + e^{\frac{1+x}{2}}\right)$, with exact solutions $u_1(x) = e^{-\frac{1+x}{2}}$, $u_2(x) = 2\sin\frac{1+x}{2}$.

To solve a system of m linear Volterra integral equations (VIEs), one eventually needs to handle a linear system of size $m(N+1) \times m(N+1)$, where N denotes the degree of the Jacobi polynomials employed in the approximation. Based on the results presented in Table 1 of Example 4.1, it is observed that the iterated JSG method can achieve an accuracy of up to 10^{-5} in both the infinity norm and the L^2-norm by solving a system of size 8×8 (i.e., $N = 3$). In contrast, the JSG method requires a system of size 10×10 (i.e., $N = 4$) to reach a similar level of precision. Moreover, to achieve a higher accuracy of 10^{-10} in the infinity norm using the iterated JSG method, the system size must be increased to 16×16, corresponding to $N = 7$. Likewise, a system of the same size is necessary to attain an accuracy of 10^{-11} in the weighted L^2-norm. These observations highlight the effectiveness of the JSIG method in delivering high accuracy with comparatively smaller system dimensions.

Table 1. JSG and iterated JSG Methods

N	$\|U_N - Z\|_\infty$	a_1	$\|U_N - U\|$	$\bar{a}_1$	$\left\|\tilde{U}_N - U\right\|_\infty$	b_1	$\left\|\tilde{U}_N - \mathbf{U}\right\|$	$\bar{b}_1$
2	1.635059×10^{-2}	5.93	7.813848×10^{-3}	6.99	2.413050×10^{-3}	8.69	1.538814×10^{-3}	9.34
3	7.416464×10^{-4}	6.56	2.713087×10^{-4}	7.48	9.564959×10^{-5}	8.42	6.046309×10^{-5}	8.84
4	5.932568×10^{-5}	7.02	2.453607×10^{-5}	7.66	1.848857×10^{-6}	9.52	1.184810×10^{-6}	9.84
5	1.357986×10^{-6}	8.39	5.633784×10^{-7}	8.94	8.133696×10^{-8}	10.14	5.412656×10^{-8}	10.40
6	1.047688×10^{-7}	8.97	3.685340×10^{-8}	9.55	4.501508×10^{-9}	10.73	2.962201×10^{-9}	10.96
7	2.084666×10^{-9}	10.27	6.326940×10^{-10}	10.88	1.092350×10^{-10}	11.79	7.298220×10^{-11}	11.99

Example 4.2. For the WS kernels we take the system of linear second kind VIE as given below

$$y_1(t) - \int_0^t (t-v)^{-1/2} y_1(v) dv + \int_0^t (t-v)^{-1/2} v^6 y_2(v) dv = g_1(t),$$

$$y_2(t) - \int_0^t (t-v)^{-1/2} 2 y_1(v) dv + \int_0^t (t-v)^{-1/2} v y_2(v) dv = g_2(t),$$

where $g_1(t) = t^{\frac{15}{2}} \left[\frac{4096}{6435} + \sqrt{t} - \frac{65536}{109395} t \right]$, $g_2(t) = t^7 \left[1 - \frac{65536}{109395} \sqrt[3]{t} \right]$ and the exact solutions are $y_1(t) = t^8$, $y_2(t) = t^7$. After transformations, the above system becomes

$$u_1(x) - \int_{-1}^1 \left(\frac{1+x}{4} \right)^{\frac{1}{2}} (1-\theta)^{-\frac{1}{2}} u_1(s(x,\theta)) d\theta + \int_{-1}^1 \left(\frac{1+x}{4} \right)^{\frac{1}{2}} (1-\theta)^{-\frac{1}{2}} \left(\frac{1+x}{2} \right)^6 \left(\frac{1+\theta}{2} \right)^6 u_2(s(x,\theta)) d\theta = f_1(x),$$

$$u_2(x) - \int_{-1}^1 2 \left(\frac{1+x}{4} \right)^{\frac{1}{2}} (1-\theta)^{-\frac{1}{2}} u_1(s(x,\theta)) d\theta + \int_{-1}^1 \left(\frac{1+x}{4} \right)^{\frac{1}{2}} (1-\theta)^{-\frac{1}{2}} \left(\frac{1+x}{2} \right) \left(\frac{1+\theta}{2} \right) u_2(s(x,\theta)) d\theta = f_2(x)$$

where $f_1(x) = \left(\frac{1+x}{2} \right)^{\frac{15}{2}} \left[\frac{4096}{6435} + \sqrt{\frac{1+x}{2}} - \frac{65536}{109395} \left(\frac{1+x}{2} \right) \right]$, $f_2(x) = \left(\frac{1+x}{2} \right)^7 \left[1 - \frac{65536}{109395} \sqrt[3]{\frac{1+x}{2}} \right]$ with the exact solutions $u_1(x) = \left(\frac{1+x}{2} \right)^8$, $u_2(x) = \left(\frac{1+x}{2} \right)^7$.

From Table 2 in Example 4.2, it is clear that the JSG method needs to solve a system of size 12×12 (i.e., $N = 5$) to reach an accuracy level of 10^{-3} in both the infinity norm and the L^2-norm. On the other hand, iterated JSG method achieves the same accuracy using a slightly smaller system of size 10×10 (i.e., $N = 4$). Additionally, when aiming for a higher accuracy of 10^{-4}, the iterated JSG approach requires solving a system of size 14×14, corresponding to $N = 6$. This indicates that iterated JSG can deliver better precision with fewer degrees of freedom compared to the standard JSG method.

Example 4.3. For the mixed type kernels we take the system of linear second kind VIE as given below

$$y_1(t) - \int_0^t \frac{1}{100} (t-v)^{-1/2} y_1(v) dv + \int_0^t v^5 y_2(v) dv = g_1(t),$$

$$y_2(t) - \int_0^t \frac{1}{500} (t-v)^{-1/2} 2 y_1(v) dv + \int_0^t v^7 y_2(v) dv = g_2(t),$$

TABLE 2. JSG and iterated JSG Methods

N	$\|\mathbf{U}_N - \mathbf{U}\|_\infty$	a_1	$\|\mathbf{U}_N - \mathbf{U}\|$	$\bar{a}_1$	$\left\|\tilde{U}_N - \mathbf{U}\right\|_\infty$	b_1	$\left\|\tilde{U}_N - \mathbf{U}\right\|$	$\bar{b}_1$
2	3.682644×10^{-1}	1.44	1.694650×10^{-1}	2.56	1.541739×10^{-1}	2.70	1.332672×10^{-1}	2.91
3	1.432558×10^{-1}	1.77	4.914982×10^{-2}	2.74	4.285625×10^{-2}	2.87	3.523993×10^{-2}	3.05
4	4.134446×10^{-2}	2.30	1.503026×10^{-2}	3.03	9.971044×10^{-3}	3.32	8.261356×10^{-3}	3.46
5	8.277467×10^{-3}	2.98	3.215109×10^{-3}	3.57	2.553927×10^{-3}	3.71	2.304511×10^{-3}	3.77
6	1.079621×10^{-3}	3.81	4.293593×10^{-4}	4.33	3.175799×10^{-4}	4.50	3.067567×10^{-4}	4.51
7	8.084161×10^{-5}	4.84	2.703559×10^{-5}	5.41	2.133803×10^{-5}	5.53	1.916550×10^{-5}	5.58

where $g_1(t) = t^8 + \frac{t^{13}}{13} + -\frac{16384}{2734875}t^{\frac{17}{2}}$ and $g_2(t) = t^7 + \frac{t^{15}}{15} + -\frac{16384}{13674375}t^{\frac{17}{2}}$ the exact solutions are $y_1(t) = t^8, y_2(t) = t^7$. and the exact solutions are $y_1(t) = t^8, y_2(t) = t^7$. After transformations, the above system becomes

$$u_1(x) - \int_{-1}^{1} \frac{1}{100}\left(\frac{1+x}{4}\right)^{\frac{1}{2}}(1-\theta)^{-\frac{1}{2}}u_1(s(x,\theta))d\theta + \int_{-1}^{1}\left(\frac{(1+x)^{\frac{11}{2}}}{2^6}\right)\left(\frac{1+\theta}{2}\right)^5 u_2(s(x,\theta))d\theta = f_1(x),$$

$$u_2(x) - \int_{-1}^{1} \frac{1}{500}\left(\frac{1+x}{4}\right)^{\frac{1}{2}}(1-\theta)^{-\frac{1}{2}}u_1(s(x,\theta))d\theta + \int_{-1}^{1}\left(\frac{(1+x)^{\frac{15}{2}}}{2^8}\right)\left(\frac{1+\theta}{2}\right)^7 u_2(s(x,\theta))d\theta = f_2(x),$$

where $f_1(x) = \left(\frac{1+x}{2}\right)^8 + \frac{\left(\frac{1+x}{2}\right)^{13}}{13} + -\frac{16384}{2734875}\left(\frac{1+x}{2}\right)^{\frac{17}{2}}$ and $f_2(x) = \left(\frac{1+x}{2}\right)^7 + \frac{\left(\frac{1+x}{2}\right)^{15}}{15} + -\frac{16384}{13674375}\left(\frac{1+x}{2}\right)^{\frac{17}{2}}$

As shown in Table 3 of Example 4.3, to achieve an accuracy of 10^{-3} in both the infinity norm and the L^2-norm, the JSG method needs to solve a linear system of size 12×12, which corresponds to $N = 5$. In comparison, the iterated JSG method reaches the same level of accuracy in the uniform norm using a smaller system of size 8×8 (i.e., $N = 3$). Furthermore, if the desired accuracy is increased to 10^{-4} in the uniform norm, the iterated JSG method requires solving a system of size 10×10, which corresponds to $N = 4$. These findings suggest that the iterated JSG approach is more efficient in terms of system size when aiming for higher precision.

TABLE 3. JSG and iterated JSG Methods

N	$\|\mathbf{U}_N - \mathbf{U}\|_\infty$	a_1	$\|\mathbf{U}_N - \mathbf{U}\|$	$\bar{a}_1$	$\left\|\tilde{U}_N - \mathbf{U}\right\|_\infty$	b_1	$\left\|\tilde{U}_N - \mathbf{U}\right\|$	$\bar{b}_1$
3	1.432608×10^{-1}	1.76	4.851041×10^{-2}	2.75	2.0262598×10^{-3}	5.64	8.6016124×10^{-1}	6.42
4	4.392670×10^{-2}	2.25	1.492509×10^{-2}	3.03	4.082447×10^{-4}	5.62	1.575835×10^{-1}	6.31
5	9.372161×10^{-3}	2.90	3.1981402×10^{-3}	3.56	5.097686×10^{-5}	6.14	1.855500×10^{-2}	6.76
6	1.244697×10^{-3}	3.73	4.264152×10^{-4}	4.33	2.824300×10^{-6}	7.13	1.416997×10^{-3}	7.51
7	7.780468×10^{-5}	4.86	2.665097×10^{-5}	5.41	2.018383×10^{-7}	7.92	9.680213×10^{-5}	8.29
8	1.106147×10^{-7}	7.70	3.698592×10^{-8}	8.22	1.106119×10^{-7}	7.70	3.698592×10^{-5}	8.22

4.1. **Conclusion.** We achieved the superconvergence result and in order to approximate the system of linear VIE for both smooth and WS kernels, we created the JSG and its iterated variant. We have demonstrated that the JSIG technique offers a superior approximation than the JSG method.

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