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METHODOLOGY TO COMPUTE COUPLED LOWER AND UPPER SOLUTIONS FOR REACTION DIFFUSION EQUATIONS

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ABSTRACT. The applications of reaction-diffusion equations can be seen in various branches of science and engineering. Here we have used the generalized monotone method to construct the solution of a reaction-diffusion equation under initial and boundary conditions, where the forcing function is the sum of increasing and decreasing functions. The monotone sequences obtained by the application of generalized monotone method coupled with coupled lower and upper solutions, converge uniformly and monotonically to coupled minimal and maximal solutions. Not only are the lower and upper solutions relatively easy to compute, they also guarantee the interval of existence. Considering these lower and upper solutions as the initial approximation, we develop a method to compute a sequence of coupled lower and upper solutions on the interval of existence or on any desired interval of existence. The coupled minimal and maximal solutions converge to the unique solution of the reaction diffusion equation if the uniqueness conditions are satisfied. We will also provide some numerical results to support our methodology.

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1. Introduction

In the recent past the study of nonlinear reaction diffusion equations has gained much interest due to it's applications in different streams of research, like science, engineering, finance, economics etc., ([1] , [2]) . There are various numerical methods available to solve these reaction-diffusion equations, ([3] , [4] , [5] , [6]). In this paper, we have used the monotone method ([7] , [8]) using the lower and upper solutions. The monotone method yields monotone sequences which converge to minimal and maximal solutions of the corresponding reaction-diffusion equation. Not only is the method theoretical and computational, it also guarantees the interval of existence by using lower and upper solutions. In the case if the forcing function is increasing or can be

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made as an increasing function, we get increasing sequences of approximate solutions, which converge to the minimal solution. If the forcing function is decreasing, then starting from the lower solution, with additional assumptions the monotone method will yield intertwined alternating sequences. In general the forcing function is a sum of increasing and decreasing functions. In such cases the generalized monotone method along with coupled lower and upper solutions will be a suitable choice to solve the non-linear problem ([9] , [10] , [11]). The theoretical results obtained by generalized monotone method can be justified by the numerical results. The iterates are solutions of simple linear equations. The generalized monotone method can be developed using the natural lower and upper solutions as well as coupled lower and upper solutions of simple linear equations ([7], [12]). Natural lower and upper solutions can be obtained easily by finding the equilibrium solutions and these solutions act as natural lower and upper solutions. Consideration of natural lower and upper solutions to obtain the successive iterates needs extra assumptions. In this case, although it guarantees the existence of solution on a limited interval, it doesn't guarantee the existence of the solution on the desired interval. On the other hand, if coupled lower and upper solutions of Type I are used, we don't need extra assumptions and the interval of existence is still guaranteed, as the coupled lower and upper solutions of Type I imply that they are natural lower and upper solutions as well. In this paper, we recall the generalized monotone method developed for reaction-diffusion equations. Also we provide a methodology to compute lower and upper solutions using natural lower and upper solutions as initial approximations. This will provide the coupled lower and upper solution on certain limited interval. Using the generalized monotone approach, we can continue the iterative process until we compute the coupled lower and upper solution for the desired interval. If further uniqueness conditions are satisfied, then we can compute the unique solution of the reaction-diffusion equation. We provide a numerical result to support our analytical results.

2. **Preliminary Results**

In this section, we recall known results related to reaction diffusion equation of the form:

$$\begin{aligned} u_t - u_{xx} &= f(t,x,u) + g(t,x,u), \\ u(0,x) &= h(x) \quad \text{on} \quad [0,L], \\ u(t,0) &= u(t,L) = 0 \end{aligned} \tag{2.1}$$

where $h(x)$ is $C[0,L]$ with $h(0) = h(L) = 0$, $f, g \in C(J \times \mathbb{R}^2, \mathbb{R}^2)$, where $J = [0,T]$.

Here and throughout this paper we assume $f(t,x,u)$ is non-decreasing in u and $g(t,x,u)$ is non-increasing in u and for (t,x) on $(J \times [0,L])$.

We recall the following known definitions which are needed for our main results.

Definition 2.1.

The functions $v, w \in C^{1,2}(J \times [0, L], \mathbb{R})$ are called natural lower and upper solutions of (2.1) if they satisfy the following inequalities:

$$v_t - v_{xx} \leq f(t, x, v) + g(t, x, v),\ v(0, x) \leq h(x), v(t, 0) = 0, v(t, L) = 0$$

and

$$w_t - w_{xx} \geq f(t, x, w) + g(t, x, w),\ w(0, x) \geq h(x), w(t, 0) = 0, w(t, L) = 0$$

Definition 2.2.

The functions $v, w \in C^{1,2}(J \times [0, L], \mathbb{R})$ are called coupled lower and upper solutions of Type I of (2.1) if they satisfy the following inequalities:

$$v_t - v_{xx} \leq f(t, x, v) + g(t, x, w),\ v(0, x) \leq h(x), v(t, 0) = 0, v(t, L) = 0$$

and

$$w_t - w_{xx} \geq f(t, x, w) + g(t, x, v),\ w(0, x) \geq h(x), w(t, 0) = 0, w(t, L) = 0$$

The below result is the existence theorem for the equation (2.1).

Theorem 2.3. *Assume that*
(i) $v_0, w_0 \in C^{1,2}(J \times [0, L], \mathbb{R})$ are coupled lower and upper solutions of Type I of (2.1) with $v_0(t, x) \leq w_0(t, x)$ on $(J \times [0, L])$.
(ii) $f, g \in C[J \times \mathbb{R}^2, \mathbb{R}^2]$, $f(t, x, u)$ is non-decreasing in u and $g(t, x, u)$ is non-increasing in u on $(J \times [0, L])$.
Then there exists monotone sequences $v_n(t, x)$ and $w_n(t, x)$ on $(J \times [0, L])$ such that $v_n(t, x) \longrightarrow v(t, x)$ and $w_n(t, x) \longrightarrow w(t, x)$ uniformly and monotonically and (v, w) are coupled minimal and maximal solutions, respectively of equation (2.1). That is (v, w) satisfy:

(2.2)
$$v_t - v_{xx} = f(t, x, v) + g(t, x, w), v(0, x) = h(x), v(t, 0) = v(t, L) = 0\, on\, (J \times [0, L])$$

(2.3)
$$w_t - w_{xx} = f(t, x, w) + g(t, x, v), w(0, x) = h(x), w(t, 0) = w(t, L) = 0\, on\, (J \times [0, L])$$

Here the iterative scheme is given by

(2.4)
$$(v_{n+1})_t - (v_{n+1})_{xx} = f(t, x, v_n) + g(t, x, w_n), v_{n+1}(0, x) = h(x), v_{n+1}(t, 0) = v_{n+1}(t, L) = 0\, on\, (J \times [0, L])$$

(2.5)
$$(w_{n+1})_t - (w_{n+1})_{xx} = f(t, x, w_n) + g(t, x, v_n), w_{n+1}(0, x) = h(x), w_{n+1}(t, 0) = w_{n+1}(t, L) = 0\, on\, (J \times [0, L])$$

Further if u is any solution of the equation (2.1) such that $v_0 \leq u_0 \leq w_0$ then $v \leq u \leq w$ on $(J \times [0, L])$.

Proof. The proof of the above Theorem 2.3 is on the same lines with the proof provided in [13]. □

The following result is the comparison result related to the coupled lower and upper solution.

Theorem 2.4. *Let all the hypothesis of Theorem 2.3 be satisfied. Further let $f(t,x,u)$ and $g(t,x,u)$ satisfy the one sided Lipschitz condition of the form:*

$$f(t,x,u) - f(t,x,\overline{u}) \leq L(u - \overline{u}),\ L > 0,$$

$$g(t,x,u) - g(t,x,\overline{u}) \geq M(u - \overline{u}),\ M > 0$$

whenever $u \geq \overline{u}$. Then $v(t,x) = w(t,x) = u(t,x)$, where $u(t,x)$ is the unique solution of (2.1).

Proof. The proof of the above Theorem 2.4 is on the same lines as the proof provided in [13]. □

Next we state a corollary of Theorem 2.4 which is useful in the monotone method or the generalized monotone method.

Corollary 2.5. *Let $(p(t,x))_t - (p(t,x))_{xx} \leq (L+M)p,\ L, M \geq 0$. Then we have $p(t,x) \leq 0$, whenever $p(0,x) \leq 0, p(t,0) \leq p(t,L)$.*

Proof. The proof of this corollary is on the same lines as the proof provided in [10]. □

The next result is the generalized monotone method for equation (2.1) where we use natural lower and upper solutions.

Theorem 2.6. *Assume that*
(i) $v_0, w_0 \in C^{1,2}(J \times [0,L], \mathbb{R})$ are natural lower and upper solutions of (2.1) with $v_0(t,x) \leq w_0(t,x)$ on $(J \times [0,L])$.
(ii) $f, g \in C[J \times \mathbb{R}^2, \mathbb{R}^2]$, $f(t,x,u)$ is non-decreasing in u and $g(t,x,u)$ is non-increasing in u on $(J \times [0,L])$.
Then there exists monotone sequences $v_n(t,x)$ and $w_n(t,x)$ on $(J \times [0,L])$ such that $v_n(t,x) \longrightarrow v(t,x)$ and $w_n(t,x) \longrightarrow w(t,x)$ uniformly and monotonically and (v,w) are coupled minimal and maximal solutions, respectively of equation (2.1). That is (v,w) satisfy:

(2.6)
$$v_t - v_{xx} = f(t,x,v) + g(t,x,w), v(0,x) = h(x), v(t,0) = v(t,L) = 0 \,on\, (J \times [0,L])$$

(2.7)
$$w_t - w_{xx} = f(t,x,w) + g(t,x,v), w(0,x) = h(x), w(t,0) = w(t,L) = 0 \,on\, (J \times [0,L])$$

provided $v_0 \leq v_1$ and $w_1 \leq w_0$ on $(J \times [0,L])$.

Proof. The proof of the above Theorem 2.6 is on the same lines as the proof provided in [13]. □

The above theorem uses v_0, w_0 as natural lower and upper solutions. Then v_1, w_1 will be coupled lower and upper solutions only on some interval $[0, t)$ and not necessarily on $[0, T]$. This is the motivation for our main result relative to the equation (2.1).

3. **Main Result**

The solution of the reaction-diffusion equation when the forcing function is the sum of the non-decreasing and non-increasing functions can be obtained by applying the generalized monotone method. If we use the natural lower and upper solutions, we need the extra assumption that the lower solution has to be less than the upper solution. If we use a coupled lower and upper solution of Type I, we do not need any extra assumption. But in most of the physical problems such as population models, the natural lower and upper solutions are readily available due to the fact that the equilibrium solutions are indeed the natural lower and upper solutions. Using these natural lower and upper solutions and applying Theorem 2.6, the coupled lower and upper solutions of Type I can be obtained. The first iterates thus obtained are on some interval $[0, t]$ and not necessarily on the whole interval $[0, T]$. This is the motivation for our main results. Our aim is to compute coupled lower and upper solution on the interval $[0, T]$, so that we can apply Theorem 2.3 to compute coupled minimal and maximal solutions of the equation 2.1. Further if f, g satisfy the one sided Lipschitz condition we can show that the coupled minimal and maximal solutions of the equation 2.1 will converge to an unique solution.

Theorem 3.1. *Assume that*
(i) $v_0, w_0 \in C^{1,2}[J \times [0, L], \mathbb{R}]$ *are natural lower and upper solutions of (2.1) such that* $v_0(t, x) \leq w_0(t, x)$ *on* $(J \times [0, L])$.
(ii) $f, g \in C[J \times \mathbb{R}^2, \mathbb{R}^2]$ *such that* $f(t, x, u)$ *is non-decreasing in* u *and* $g(t, x, u)$ *is non-increasing in* u *on* $(J \times [0, L])$.
Then there exists monotone sequences $\{v_n(t, x)\}$ *and* $\{w_n(t, x)\}$ *on* $(J \times [0, L])$ *such that* $v_n(t, x) \to v(t, x)$ *and* $w_n(t, x) \to w(t, x)$ *uniformly and monotonically and* (v, w) *are coupled lower and upper solutions of (2.1) such that* $v \leq w$ *on* $(J \times [0, L])$. *The iterative scheme is given by*
$(v_n)_t - (v_n)_{xx} = f(t, x, v_{n-1}) + g(t, x, w_{n-1}),\ v_n(0, x) = u(0, x)$ *on* $[0, t_n]$
$(w_n)_t - (w_n)_{xx} = f(t, x, w_{n-1}) + g(t, x, v_{n-1}),\ w_n(0, x) = u(0, x)$ *on* $[0, \bar{t}_n]$

Proof. From Theorem 2.6 we have $v_0(t, x) \leq v_1(t, x)$ on $([0, t_1] \times [0, L])$ and $w_1(t, x) \leq w_0(t, x)$ on $([0, \bar{t}_1] \times [0, L])$.
If $t_1 \geq T$, and $\bar{t}_1 \geq T$ there is nothing to prove since one can use Theorem 2.3 to compute coupled minimal and maximal solutions. If not, certainly $t_1 < T$ and $\bar{t}_1 < T$. Then redefine $v_1(t, x)$ and $w_1(t, x)$ as follows:

$(v_1)_t - (v_1)_{xx} = f(t,x,v_0) + g(t,x,w_0),\ v_1(0,x) = u(0,x)\,\text{on}\,[0,t_1]$
$(w_1)_t - (w_1)_{xx} = f(t,x,w_0) + g(t,x,v_0),\ w_1(0,x) = u(0,x)\,\text{on}\,[0,\bar{t}_1]$
and $v_1(t,x) = v_0(t,x)$ on $[t_1,T]$, $w_1(t,x) = w_0(t,x)$ on $[\bar{t}_1,T]$, such that $v_1(t_1,x) = v_0(t_1,x), w_1(\bar{t}_1,x) = w_0(\bar{t}_1,x)$.

Proceeding in this manner, we can obtain the n^{th} elements of the sequences v_n and w_n as follows:
$(v_n)_t - (v_n)_{xx} = f(t,x,v_{n-1}) + g(t,x,w_{n-1}), v_n(0,x) = u(0,x),$ on $[0,t_n]$
$(w_n)_t - (w_n)_{xx} = f(t,x,w_{n-1}) + g(t,x,v_{n-1}), w_n(0,x) = u(0,x),$ on $[0,\bar{t}_n]$
and $v_n(t,x) = v_0(t,x)$ on $[t_n,T]$, $w_n(t,x) = w_0(t,x)$ on $[\bar{t}_n,T]$, such that $v_n(t_n,x) = v_0(t_n,x)$, $w_n(\bar{t}_n,x) = w_0(\bar{t}_n,x)$.

We have omitted the details since the procedure is very similar to ordinary differential equations as in [14].

Now let v_n, w_n intersect v_0 and w_0 at $t_n,\ \bar{t}_n$ respectively.
If $t_n \geq T$, and $\bar{t}_n \geq T$ we can stop the process. Certainly $v_n \leq w_n$ and v_n and w_n are coupled minimal and maximal solutions of the equation (2.1) respectively.
Now we can show that the sequence $\{v_n(t,x)\}$ and $\{w_n(t,x)\}$ constructed above are equicontinuous and uniformly bounded on $(J \times [0,L])$. Hence by Ascoli Arzela's theorem, a subsequence converges uniformly and monotonically. Since the sequences are monotone, the entire sequence converges uniformly and monotonically to v and w respectively. As $n \to \infty$, $\quad t_n, \bar{t}_n \to T, v_n(t,x) \to v(t,x)$, and $w_n(t,x) \to w(t,x)$, uniformly and monotonically.
Further, $v_t - v_{xx} = f(t,x,v) + g(t,x,w),\ v(0,x) = u(0,x), v(t,0) = v(t,L) = 0\, on\, (J \times [0,L])$
and $w_t - w_{xx} = f(t,x,w) + g(t,x,v),\ w(0,x) = u(0,x), w(t,0) = w(t,L) = 0\, on\, (J \times [0,L])$.

Hence v, w are coupled lower and upper solutions of the equation (2.1) on $(J \times [0,L])$. This concludes the proof. □

Theorem 3.2. *Let all the hypotheses of Theorem 3.1 hold. Then there exist sequences* $\{v_n^*\}, \{w_n^*\}$ *in* $(J \times [0,L])$*, such that they converge uniformly and monotonically to coupled minimal and maximal solutions of equation (2.1). These sequences converge at a much faster rate than the sequences of Theorem 3.1. The sequences* $\{v_n^*\}$ *and* $\{w_n^*\}$ *are developed as follows, where the iterative scheme is given by*

$$(v_{n+1}^*)_t - (v_{n+1}^*)_{xx} = f(t,x,v_n^*) + g(t,x,w_n^*), v_n(0,x) = u(0,x)$$

and

$$(w_{n+1}^*)_t - (w_{n+1}^*)_{xx} = f(t,x,w_n^*) + g(t,x,v_{n+1}^*), w_n(0,x) = u(0,x),$$

Proof. Let $v_1 = v_0^*$, then $(w_0^*)_t - (w_0^*)_{xx} = f(t,x,w_0) + g(t,x,v_0^*)$
We will prove that $w_0^* \leq w_1$ on $(J \times [0,L])$.
For that purpose, set $p(t) = w_0^* - w_1$, $p(0) = 0$.

$p'(t) = ((w_0^*)_t + (w_0^*)_{xx}) - ((w_1)_t + (w_1)_{xx}) = f(t,x,w_0) + g(t,x,v_0^*) - f(t,x,w_0) - g(t,x,v_0) = g(t,x,v_0^*) - g(t,x,v_0) \leq 0 (\because v_0^* \geq v_0)$. This proves $w_0^* \leq w_1$. Continuing this process we can show that the sequences $\{v_n^*\}$ and $\{w_n^*\}$ converges faster than the sequences $\{v_n\}$ and $\{w_n\}$ computed using Theorem 3.1. □

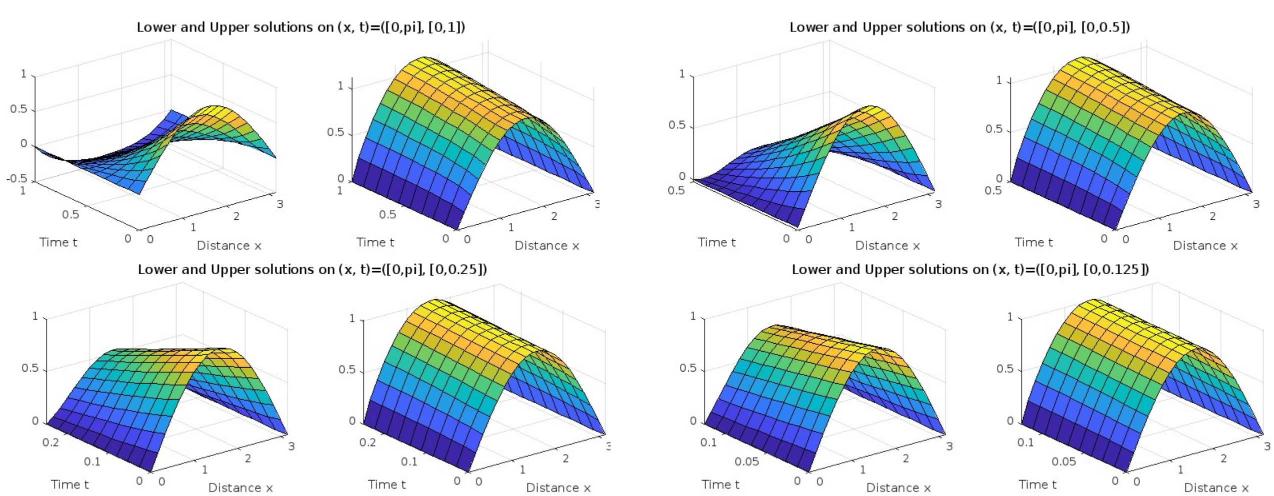
4. **Numerical Results**

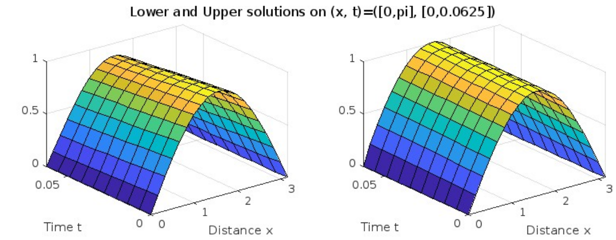
In this section we provide a numerical example to support the main results. Here we consider a simple prey-predator model with initial conditions. In order to apply Theorem 3.1 we require the natural lower and upper solutions. The equilibrium solutions of the model serve as the natural lower and upper solutions.

$$\begin{aligned} u_t - u_{xx} &= u - u^2, \\ \text{with B.C}\, u(t,0) &= u(t,\pi) = 0 \qquad (4.1) \\ \text{and I.C}\, u(0,x) &= \sin(x). \end{aligned}$$

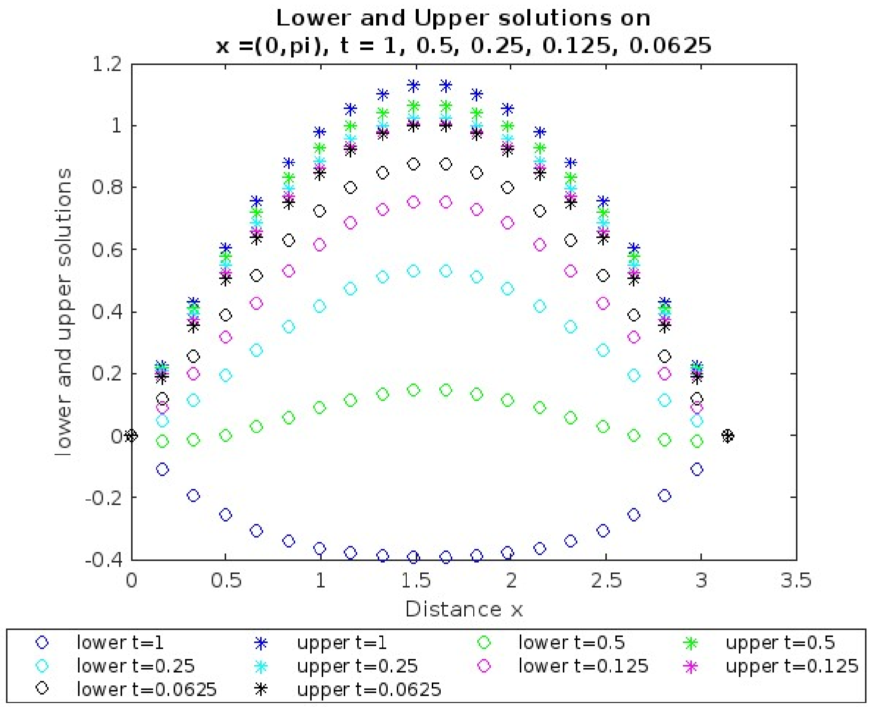
Here $f = u$ and $g = -u^2$.
Let $v_0(t,x) = 0$ and $w_0(t,x) = 1$ be the natural lower and upper solutions of 4.1.
Then, $v_{1t} - v_{1xx} = v_0 - (w_0)^2$ implies $v_{1t} - v_{1xx} = 0 - 1^2 = -1$, with B.C $v(t,0) = v(t,\pi) = 0$ and I.C $v(0,x) = \sin(x)$ and $w_{1t} - w_{1xx} = w_0 - (v_0)^2$ implies $w_{1t} - w_{1xx} = 1 - 0^2 = 1$, with B.C $w(t,0) = w(t,\pi) = 0$ and I.C $w(0,x) = \sin(x)$.

The below graphs give the first iterates on the interval $x = (0,\pi)$ and $t = (0,T)$, with $T = 1, 0.5, 0.25, 0.125, 0.0625$ respectively.

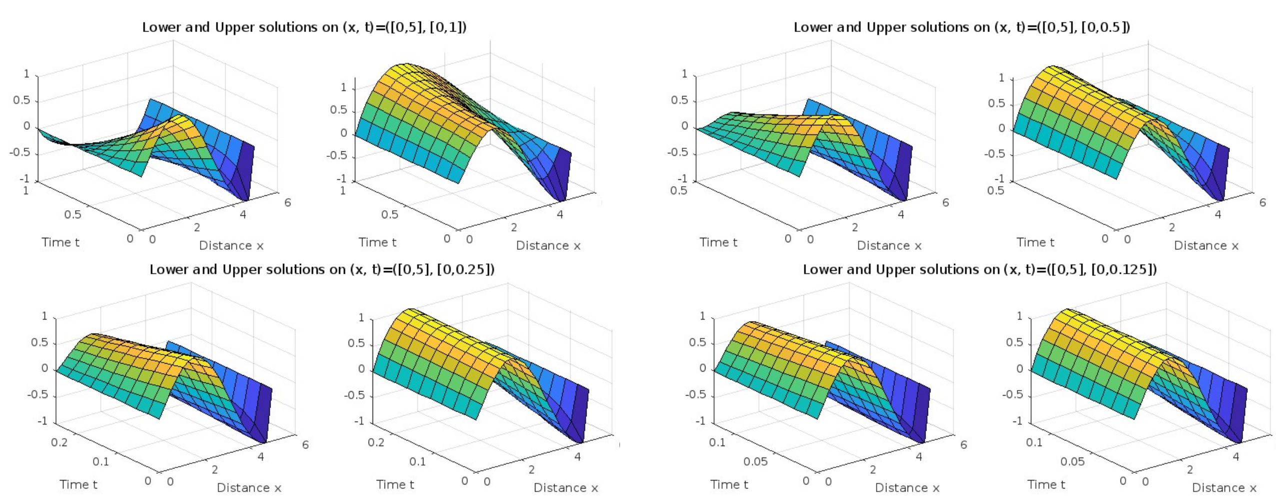


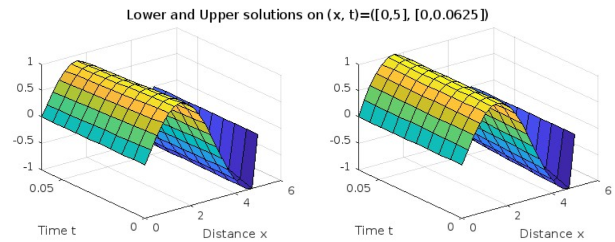


The below graph is the 2-D plot of the first iterates on the interval $x = (0, \pi)$, where the value of t is fixed as $t = 1, 0.5, 0.25, 0.125, 0.0625$.

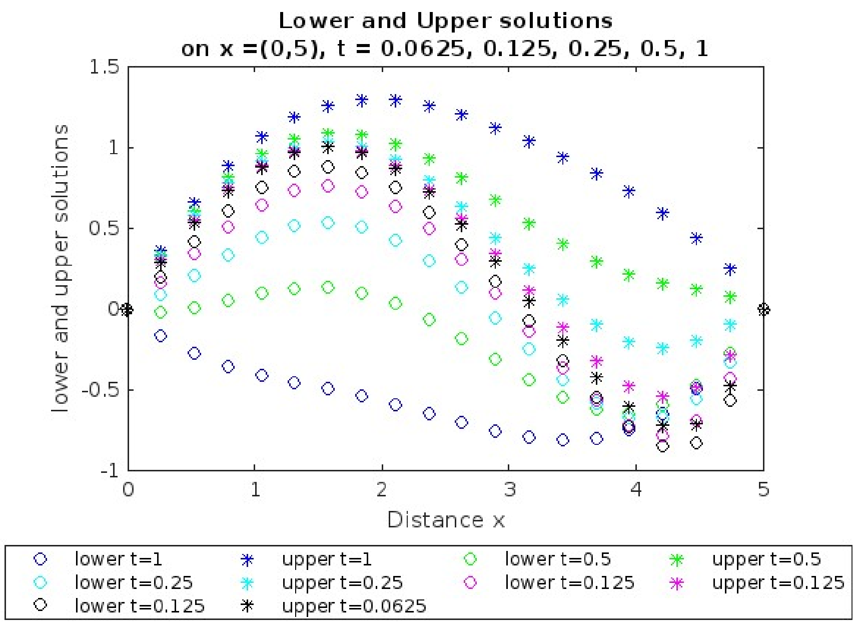


The following graphs give the first iterations in the interval $x = (0, 5)$ and $t = (0, T)$, with $T = 1, 0.5, 0.25, 0.125, 0.0625$, respectively.





The below graph is the 2-D plot of the first iterates on the interval $x = (0, 5)$, where the value of t is fixed as $t = 1, 0.5, 0.25, 0.125, 0.0625$.



The above graphs were obtained using MATLAB Code.

5. Conclusion

In this paper we have provided the theoretical results to compute the coupled lower and upper solutions on any desired interval when the natural lower and upper solutions exist on that interval. In the numerical example the first iterates are obtained for varying values of the time T and varying values of the boundary x. And we see that the lower and upper solutions converge to a unique solution on the given interval of time and on the given boundary.

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