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HALF-LINEAR THIRD-ORDER NONCANONICAL DELAY DIFFERENCE EQUATIONS: OSCILLATION VIA A SEMI-CANONICAL TRANSFORM

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ABSTRACT. In this paper, the authors derive some new criteria for the oscillation of half-linear third-order noncanonical delay difference equations. The results are obtained by first reducing the noncanonical equation into semi-canonical form. This approach reduces the set of possible nonoscillatory solutions into three classes without adding any extra conditions. Then by applying the summation averaging method and a Riccati transformation technique, they obtain new criteria for the oscillation of all solutions. Four examples illustrating the main results are provided.

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1. INTRODUCTION

We investigate the oscillatory behavior of solutions of the third-order half-linear delay difference equation

$$\Delta\left(b_2(\ell)\left(\Delta\left(b_1(\ell)\Delta x(\ell)\right)\right)^{\beta}\right)+f(\ell)x^{\beta}(\ell-\tau)=0, \tag{E}$$

where $\ell\in\mathbb{N}(\ell_0)=\{\ell_0,\ell_0+1,\cdots\}$ and ℓ_0 is a positive integer. Throughout, we assume that:

(H_1) $\{b_2(\ell)\}$, $\{b_1(\ell)\}$, and $\{f(\ell)\}$ are positive real sequences for all $\ell\in\mathbb{N}(\ell_0)$;
(H_2) τ is a positive integer and β is a ratio of odd positive integers.

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For brevity, we set

$$M_0x(\ell) = x(\ell), \quad M_1x(\ell) = b_1(\ell)\Delta x(\ell), \quad M_2x(\ell) = \alpha_2(\ell)\left(\Delta M_1 x(\ell)\right)^{\beta},$$

and

$$M_3x(\ell) = \Delta\left(M_2x(\ell)\right),$$

so that equation (E) can be written as

$$M_3x(\ell) + f(\ell)x^{\beta}(\ell-\tau) = 0.$$

In [24], the authors introduced the classification of the difference operator

$$Dz(n) = \Delta(b(n)\Delta(a(n)\Delta z(n)))$$

(also see [26] for such a classification for differential equations and [15] for dynamic equations on time scales) saying that it is in canonical form if

$$\sum_{n=n_0}^{\infty} \frac{1}{a(n)} = \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{b(n)} = \infty;$$

in noncanonical form if

$$\sum_{n=n_0}^{\infty} \frac{1}{a(n)} < \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{b(n)} < \infty;$$

and in semi-canonical form if either

$$\sum_{n=n_0}^{\infty} \frac{1}{a(n)} < \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{b(n)} = \infty$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a(n)} = \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{b(n)} < \infty.$$

As a somewhat natural extension of that scheme, we will say that equation (E), and the corresponding operator $M_3 = \Delta\left(b_2(\ell)\left(\Delta\left(b_1(\ell)\Delta x(\ell)\right)\right)^{\beta}\right)$ is in canonical form if

$$\text{(C)} \qquad \sum_{\ell=\ell_0}^{\infty} \frac{1}{b_1(\ell)} = \infty \quad \text{and} \quad \sum_{\ell=\ell_0}^{\infty} \frac{1}{b_2^{1/\beta}(\ell)} = \infty,$$

is in noncanonical form if

$$\text{(NC)} \qquad \sum_{\ell=\ell_0}^{\infty} \frac{1}{b_1(\ell)} < \infty \quad \text{and} \quad \sum_{\ell=\ell_0}^{\infty} \frac{1}{b_2^{1/\beta}(\ell)} < \infty,$$

and is in semi-canonical form if either

$$\text{(SC1)} \qquad \sum_{\ell=\ell_0}^{\infty} \frac{1}{b_1(\ell)} < \infty \quad \text{and} \quad \sum_{\ell=\ell_0}^{\infty} \frac{1}{b_2^{1/\beta}(\ell)} = \infty$$

or

$$\text{(SC2)} \qquad \sum_{n=n_0}^{\infty} \frac{1}{b_1(\ell)} = \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{b_2^{1/\beta}(\ell)} < \infty.$$

In what follows, we assume that $M_3x(\ell)$ is of noncanonical type, that is,

$$\sum_{\ell=\ell_0}^{\infty} \frac{1}{b_1(\ell)} < \infty \quad \text{and} \quad \sum_{\ell=\ell_0}^{\infty} \frac{1}{b_2^{1/\beta}(\ell)} < \infty. \tag{1.1}$$

By a *solution* of (E), we mean a real sequence $\{x(\ell)\}$ that is defined for $\ell \geq \ell_0 - \tau$ and that satisfies (E) for all $\ell \in \mathbb{N}(\ell_0)$. A nontrivial solution of (E) is said to be *oscillatory* if the terms of the sequence $\{x(\ell)\}$ are neither eventually all positive nor eventually all negative, and *nonoscillatory* otherwise.

In view of the significance of third-order difference and differential equations in the applications of real world problems, the study of the qualitative theory of such equations has gained momentum in the last three decades; see, for example the monographs [1, 2].

In particular, oscillation theory for third-order functional difference equations has received great attention in recent years as is evident from the extensive literature on this topic and as can be seen in [3–6, 8, 9, 11–25, 27] and the references contained therein. Most of these papers discuss oscillatory properties of solutions of the so called canonical type equations, namely, the condition opposite to (1.1) ((NC)) holds, namely, (C) holds; see, for example, the papers [2, 3, 6, 9, 11, 13, 14, 17–23, 25, 27] and the references therein.

In [4, 12, 15, 24], the authors considered equation (E) and its special cases when the equation in semi-canonical form, that is, (SC1) or (SC2) holds, and they obtained oscillation criteria by transforming the equation into it into a canonical type equation.

In [8] and [5], the authors studied oscillatory properties of (E) by eliminating the four possible classes for nonoscillatory solutions:

$$S_1 = \{x(\ell) : x(\ell) > 0, M_1x(\ell) < 0, M_2x(\ell) < 0, M_3x(\ell) < 0\},$$
$$S_2 = \{x(\ell) : x(\ell) > 0, M_1x(\ell) < 0, M_2x(\ell) > 0, M_3x(\ell) < 0\},$$
$$S_3 = \{x(\ell) : x(\ell) > 0, M_1x(\ell) > 0, M_2x(\ell) > 0, M_3x(\ell) < 0\},$$
$$S_4 = \{x(\ell) : x(\ell) > 0, M_1x(\ell) > 0, M_2x(\ell) < 0, M_3x(\ell) < 0\}.$$

On the other hand, the authors in [16] considered equation (E) with $\beta = 1$ and transformed it into a canonical type equation, which reduced the number of possible classes of nonoscillatory solutions to two, namely, S_2 and S_3. Oscillation criteria were then obtained by eliminating these two classes.

It is interesting to note that the operator M_3x is not linear if $\beta \neq 1$, and so it is not possible to transform the noncanonical operator M_3x into one of canonical type. Therefore, we first transform equation (E) into a semi-canonical type equation, and we are able to do this without asking any additional conditions on the coefficient functions. This reduces the number of possible classes of nonoscillatory solutions to three types; the class S_1 is automatically eliminated from the classification, again without assuming any extra conditions. Thus, the oscillation of (E) is obtained by eliminating the remaining three types of nonoscillatory solutions.

The results obtained in this paper are new and complement existing results reported in the literature. The results here are somewhat motivated by the recent results in [7, 10, 16] established for delay differential and dynamic equations on time scales. Some examples are provided to illustrate the importance and usefulness of our main results.

2. **Preliminary Results**

In view of (1.1), we can introduce the following notation:

$$B_1(\ell) = \sum_{s=\ell}^{\infty} \frac{1}{b_1(s)}, \quad d_1(\ell) = b_1(\ell)B_1(\ell)B_1(l+1), \quad d_2(\ell) = \frac{b_2(\ell)}{B_1^{\beta}(\ell+1)},$$

$$D_1(\ell) = \sum_{s=\ell_1}^{\ell-1} \frac{1}{d_1(s)}, \quad \text{and} \quad D_2(\ell) = \sum_{s=\ell}^{\infty} \frac{1}{d_2^{1/\beta}(s)}$$

where ℓ_1 is taken large enough.

We start with the following theorems.

Theorem 2.1. *The noncanonical operator $M_3x(\ell)$ has the following semi-canonical representation*

$$M_3x(\ell) = \Delta\left(\frac{b_2(\ell)}{B_1^{\beta}(\ell+1)}\left(\Delta\left(b_1(\ell)B_1(\ell)B_1(l+1)\Delta\left(\frac{x(\ell)}{B_1(\ell)}\right)\right)\right)^{\beta}\right). \tag{2.1}$$

Proof. By a direct calculation, we have

$$\begin{aligned}\Delta\left(b_1(\ell)B_1(\ell)B_1(\ell+1)\Delta\left(\frac{x(\ell)}{B_1(\ell)}\right)\right) &= \Delta\left(B_1(\ell)b_1(\ell)\Delta x(\ell) + x(\ell)\right)\\ &= B_1(\ell+1)\Delta\left(b_1(\ell)\Delta x(\ell)\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\Delta\left(\frac{b_2(\ell)}{B_1^{\beta}(\ell+1)}\left(\Delta\left(b_1(\ell)B_1(\ell)B_1\left(\ell+1\right)\Delta\left(\frac{x(\ell)}{B_1(\ell)}\right)\right)\right)^{\beta}\right)\\ &\qquad = \Delta\left(b_2(\ell)\left(\Delta\left(b_1(\ell)\Delta x(\ell)\right)\right)^{\beta}\right) = M_3x(\ell).\end{aligned}$$

To see that (2.1) is in semi-canonical form, note that

$$\sum_{\ell=\ell_0}^{\infty} \frac{B_1(\ell+1)}{b_2^{1/\beta}(\ell)} \le B_1\left(\ell_0+1\right)\sum_{\ell=\ell_0}^{\infty}\frac{1}{b_2^{1/\beta}(\ell)} < \infty \tag{2.2}$$

by (1.1), and

$$\sum_{\ell=\ell_0}^{\infty} \frac{1}{b_1(\ell)B_1(\ell)B_1(\ell+1)} = \sum_{\ell=\ell_0}^{\infty}\Delta\left(\frac{1}{B_1(\ell)}\right) = \lim_{l\to\infty}\frac{1}{B_1(\ell)} - \frac{1}{B_1\left(\ell_0\right)} = \infty.$$

This completes the proof. □

It follows from Theorem 2.1 that equation (E) can be written in the equivalent semi-canonical form as

$$\Delta\left(d_2(\ell)\left(\Delta\left(d_1(\ell)\Delta\left(\frac{x(\ell)}{B_1(\ell)}\right)\right)\right)^{\beta}\right)+f(\ell)x^{\beta}(\ell-\tau)=0,$$

where

$$\sum_{\ell=\ell_0}^{\infty}\frac{1}{d_1(\ell)}=\infty \quad\text{and}\quad \sum_{\ell=\ell_0}^{\infty}\frac{1}{d_2^{1/\beta}(\ell)}<\infty.$$

Letting $\phi(\ell)=\frac{x(\ell)}{B_1(\ell)}$ and using the notation

$$F(\ell)=f(\ell)B_1^{\beta}(\ell-\tau), \tag{2.3}$$

the following results are immediate.

Theorem 2.2. *The noncanonical difference equation (E) possesses a solution $\{x(\ell)\}$ if and only if the semi-canonical equation*

$$\Delta\left(d_2(\ell)(\Delta(d_1(\ell)\Delta\phi(\ell)))^{\beta}\right)+F(\ell)\phi^{\beta}(\ell-\tau)=0 \tag{E_s}$$

has a solution $\phi(\ell)=\{\frac{x(\ell)}{B_1(\ell)}\}$.

The following corollary is immediate

Corollary 2.3. *The noncanonical difference equation* (E) *has an eventually positive solution if and only if the semi-canonical equation (E_s) has an eventually positive solution.*

Corollary 2.3 simplifies the investigation of (E) since for (E_s), we only need to deal with three classes of an eventually positive solutions, namely:

$$N_0=\{\phi(\ell)>0, L_1\phi(\ell)<0, L_2\phi(\ell)>0, L_3\phi(\ell)<0\},$$
$$N_1=\{\phi(\ell)>0, L_1\phi(\ell)>0, L_2\phi(\ell)<0, L_3\phi(\ell)<0\}$$
$$N_2=\{\phi(\ell)>0, L_1\phi(\ell)>0, L_2\phi(\ell)>0, L_3\phi(\ell)<0\}$$

where

$$L_1\phi(\ell)=d_1(\ell)\Delta\phi(\ell),\quad L_2\phi(\ell)=d_2(\ell)(\Delta L_1\phi(\ell))^{\beta},\quad\text{and}\quad L_3\phi(\ell)=\Delta(L_2\phi(\ell)).$$

For the verification of this classifications, we refer the reader to [24].

We begin with a theorem ensuring the nonexistence of solutions of type N_2.

Theorem 2.4. *Let $\{\phi(\ell)\}$ be an eventually positive solution of (E_s). If*

$$\sum_{\ell=\ell_0}^{\infty}F(\ell)D_1^{\beta}(\ell-\tau)=\infty, \tag{2.4}$$

then the class N_2 is empty.

Proof. Assume, to the contrary, that $N_2 \neq \emptyset$, say $\phi(\ell) \in N_2$. Choose $\ell_1 \in \mathbb{N}(\ell_0)$ such that $\phi(\ell-\tau) > 0$ for all $\ell \geq \ell_1$. Since $d_1(\ell)\Delta\phi(\ell)$ is positive and increasing, there is an integer $\ell_2 \in \mathbb{N}(\ell_1)$ such that

$$d_1(\ell)\Delta\phi(\ell) \geq d_1(\ell_2)\Delta\phi(\ell_2) = c > 0, \quad \text{for } \ell \geqslant \ell_2.$$

Dividing the last inequality by $d_1(\ell)$ and summing the result from ℓ_2 to $\ell-1$ gives

$$\phi(\ell) \geq c\, D_1(\ell). \tag{2.5}$$

Summing (E_s) from ℓ_2 to $\ell-1$ and using (2.5), we obtain

$$L_2\phi(\ell) = L_2\phi(\ell_2) - c^\beta \sum_{s=\ell_2}^{\ell-1} F(s) D_1^\beta(s-\tau) \to -\infty \text{ as } \ell \to \infty,$$

which contradicts the positivity of $L_2\phi(\ell)$ and proves the theorem. □

We will find the next lemma to be useful.

Lemma 2.5. *Let $\{\phi(\ell)\}$ be an eventually positive increasing solution of (E_s). If*

$$\sum_{\ell=\ell_0}^{\infty} \frac{1}{d_2^{1/\beta}(\ell)} \left(\sum_{s=\ell_0}^{\ell-1} F(s) D_1^\beta(s-\tau) \right)^{1/\beta} = \infty, \tag{2.6}$$

then $\{\phi(\ell)\}$ belongs to the class N_1 for all $\ell \geq \ell_1 \in \mathbb{N}(\ell_0)$ and

$$\phi(\ell) \geq D_1(\ell) L_1\phi(\ell) \tag{2.7}$$

for all $\ell \geq \ell_2 \in \mathbb{N}(\ell_1)$.

Proof. Since $\{\phi(\ell)\}$ is an increasing sequence we see that $\phi(\ell) \in N_1 \cup N_2$ for all $\ell \geq \ell_1 \in \mathbb{N}(\ell_0)$ where $\phi(\ell-\tau) > 0$ for all $\ell \geq \ell_1$. In view of (2.2), we see that (2.6) implies that (2.4) holds. Hence, by Theorem 2.4, $\phi(\ell) \in N_1$ for all $\ell \geq \ell_1$. Since $L_1\phi(\ell)$ is positive and decreasing, it follows that

$$\phi(\ell) = \phi(\ell_1) + \sum_{s=\ell_1}^{\ell-1} \frac{d_1(s)\Delta\phi(s)}{d_1(\ell)} \geqslant D_1(\ell) L_1\phi(\ell),$$

which completes the proof. □

Our next four theorems give conditions under which the set $N_1 \cup N_2$ is empty

Theorem 2.6. *Let $\{\phi(\ell)\}$ be an eventually positive solution of (E_s). If*

$$\sum_{\ell=\ell_0}^{\infty} \left(\frac{1}{d_2(\ell)} \sum_{s=\ell_0}^{\ell-1} F(s) \right)^{1/\beta} = \infty, \tag{2.8}$$

then $N_1 \cup N_2 = \emptyset$.

Proof. Assume that $N_1 \cup N_2$ is not empty. Choose $\ell_1 \in \mathbb{N}(\ell_0)$ such that $\phi(\ell) > 0$ and $\phi(\ell - \tau) > 0$ with $\phi(\ell) \in N_1 \cup N_2$ for all $\ell \geq \ell_1$. Since $\phi(\ell)$ is increasing, there exists $\ell_2 \in \mathbb{N}(\ell_1)$ such that $\phi(\ell) \geq \phi(\ell_1) = c > 0$ for all $\ell \geq \ell_2$. Summing (E_s) from ℓ_2 to $\ell - 1$, we obtain

$$L_2\phi(\ell) = L_2\phi(\ell_2) - \sum_{s=\ell_2}^{\ell-1} F(s)\phi^\beta(s-\tau) \leq L_2\phi(\ell_2) - c^\beta \sum_{s=\ell_2}^{\ell-1} F(s). \tag{2.9}$$

From (2.2) and (2.8) we see that

$$\sum_{\ell=\ell_0}^{\infty} F(\ell) = \infty. \tag{2.10}$$

If $\phi(\ell) \in N_2$, then (2.9) and (2.10) imply that $L_2\phi(\ell)$ is negative, which is a contradiction.

On the other hand, if $\phi(\ell) \in N_1$, then using the fact that $L_2\phi(\ell) < 0$ in (2.9), we see that

$$\Delta(L_1\phi(\ell)) \leq -\frac{c}{d_2^{1/\beta}(\ell)} \left(\sum_{s=\ell_2}^{\ell-1} F(s) \right)^{\frac{1}{\beta}}. \tag{2.11}$$

Summing (2.11) from ℓ_2 to $\ell - 1$ gives

$$L_1\phi(\ell) \leq L_1\phi(\ell_2) - c \sum_{s=\ell_2}^{\ell-1} \left(\frac{1}{d_2(s)} \sum_{j=\ell_2}^{s-1} F(j) \right)^{\frac{1}{\beta}},$$

which in view of (2.8) contradicts the positivity of $L_1\phi(\ell)$. This completes the proof. □

The following result is based on a comparison with a first-order delay difference inequality.

Theorem 2.7. *Let $\{\phi(\ell)\}$ be an eventually positive solution of (E_s). If*

$$\liminf_{\ell\to\infty} \sum_{s=\ell-\tau}^{\ell-1} \left(\frac{1}{d_2^{1/\beta}(s)} \sum_{j=\ell_0}^{s-1} F(j) D_1^\beta(j-\tau) \right)^{1/\beta} > \left(\frac{\tau}{\tau+1} \right)^{\tau+1}, \tag{2.12}$$

then the class $N_1 \cup N_2$ is empty.

Proof. Assume that $\{\phi(\ell)\}$ is a positive solution of (E_s) with $\phi(\ell) \in N_1 \cup N_2$ for all $\ell > \ell_1$. It is easy to see that (2.12) implies condition (2.6) must hold. Therefore, by Lemma 2.5, we conclude that $\phi(\ell) \in N_1$ and (2.7) holds. From (2.7), we have

$$\phi(\ell - \tau) \geq D_1(\ell-\tau) L_1\phi(\ell-\tau) \tag{2.13}$$

for all $\ell \geq \ell_2 \geq \ell_1$, for some $\ell_2 \in \mathbb{N}(\ell_1)$. Using (2.13) in (E_s), we obtain

$$-L_3\phi(\ell) \geq F(\ell) D_1^\beta(\ell-\tau)(L_1\phi(\ell-\tau))^\beta.$$

Summing the last inequality from ℓ_2 to $\ell - 1$ and using the fact that $L_1\phi(\ell)$ is decreasing, we obtain

$$-L_2\phi(\ell) \geq \sum_{s=\ell_2}^{\ell-1} F(\ell)D_1^\beta(s-\tau)(L_1\phi(s-\tau))^\beta,$$

or

$$-d_2(\ell)(\Delta(L_1\phi(\ell)))^\beta \geq (L_1\phi(\ell-\tau))^\beta \sum_{s=\ell_2}^{\ell-1} F(s)D_1^\beta(s-\tau). \tag{2.14}$$

Letting $w(\ell) = L_1\phi(\ell)$ in (2.14), we see that $\{w(\ell)\}$ is a positive solution of the first-order delay difference inequality

$$\Delta w(\ell) + \frac{1}{d_2^{1/\beta}(\ell)} \left(\sum_{s=\ell_2}^{\ell-1} F(s)D_1^\beta(s-\tau)\right)^{1/\beta} w(\ell-\tau) \leq 0. \tag{2.15}$$

An application of Lemma 6.1.6 in [2] shows that (2.15) cannot have a positive solution if (2.12) holds. This contradiction completes the proof of the theorem. □

Note that the approach we used in Theorem 2.7 requires that τ is a positive integer. However, the next result applies if τ is only nonnegative.

Theorem 2.8. *Assume that* (2.6) *holds. If*

$$\limsup_{\ell\to\infty} D_2^\beta(\ell) \sum_{s=\ell_0}^{\ell-1} F(s)D_1^\beta(s-\tau) > 1, \tag{2.16}$$

then $N_1 \cup N_2$ *is empty.*

Proof. Assume that $\phi(\ell) \in N_1 \cup N_2$ for all $\ell \geq \ell_1$. First note that $\lim_{\ell\to\infty} D_2(\ell) = 0$, which together with (2.16) implies (2.4) holds. By Lemma 2.5, $\phi(\ell)$ belongs to the class N_1 and (2.7) holds for all $\ell \geq \ell_2 \in \mathbb{N}(\ell_1)$.

Proceeding as in the proof of Theorem 2.7, we are again led to (2.14). On the other hand, note that

$$d_1(\ell)\Delta\phi(\ell) \geq -\sum_{s=\ell}^{\infty} \frac{d_2^{1/\beta}(s)}{d_2^{1/\beta}(s)}\Delta(d_1(s)\Delta\phi(s)) \geq -D_2(\ell)(L_2\phi(\ell))^{1/\beta}. \tag{2.17}$$

Using (2.17) in (2.14), we see that

$$\begin{aligned}
-L_2\phi(\ell) &\geq (L_1\phi(\ell-\tau))^\beta \sum_{s=\ell_2}^{\ell-1} F(s)D_1^\beta(s-\tau) \\
&\geq (L_1\phi(\ell))^\beta \sum_{s=\ell_2}^{\ell-1} F(s)D_1^\beta(s-\tau) \\
&\geq -L_2\phi(\ell)D_2^\beta(\ell) \sum_{s=\ell_2}^{\ell-1} F(s)D_1^\beta(s-\tau).
\end{aligned}$$

Therefore,

$$1 \geq D_2^\beta(\ell) \sum_{s=\ell_2}^{\ell-1} F(s) D_1^\beta(s-\tau),$$

which contradicts (2.16), and proves the theorem. □

The last of our theorems showing that $N_1 \cup N_2 = \emptyset$, employs the use of a Riccati transformation.

Theorem 2.9. *Assume that* (2.6) *holds. If there exists a positive nondecreasing sequence* $\{\rho(\ell)\}$ *such that*

(2.18)

$$\limsup_{\ell\to\infty} \sum_{s=\ell_2}^{\ell} \left[\frac{\rho(s)}{D_1(s-\tau) d_2^{1/\beta}(s)} \left(\sum_{j=\ell_1}^{s-1} F(j) D_1^\beta(j-\tau) \right)^{1/\beta} - \frac{(\Delta\rho(s))^2 d_1(s-\tau)}{4\rho(s)} \right] = \infty$$

for any $\ell_2 \geq \ell_1 \in \mathbb{N}(\ell_0)$, *then* $N_1 \cup N_2$ *is empty.*

Proof. Assume that $\{\phi(\ell)\}$ is a positive solution of (E_s) such that $\phi(\ell) \in N_1 \cup N_2$ for $\ell \geq \ell_1$. By Lemma 2.5, we conclude that $\phi(\ell)$ belongs to the class N_1 and (2.7) holds for all $\ell \geq \ell_2 \geq \ell_1$. From (2.7), we see that

$$\Delta \left(\frac{\phi(\ell)}{D_1(\ell)} \right) = \frac{D_1(\ell) d_1(\ell) \Delta\phi(\ell) - \phi(\ell)}{D_1(\ell) D_1(\ell+1) d_1(\ell)} \leq 0$$

and so

$$\frac{\phi(\ell)}{D_1(\ell)} \text{ is decreasing for all } \ell \geq \ell_2. \tag{2.19}$$

Define the Riccati type sequence

$$\psi(\ell) = \rho(\ell) \frac{d_1(\ell) \Delta\phi(\ell)}{\phi(\ell-\tau)} > 0, \quad \ell \geq \ell_2.$$

Then using the monotonicity of $L_1\phi(\ell)$,

$$\begin{aligned} \Delta\psi(\ell) &= \frac{\Delta\rho(\ell)}{\rho(\ell+1)} \psi(\ell+1) + \frac{\rho(\ell) \Delta(L_1\phi(\ell))}{\phi(\ell-\tau)} - \frac{\rho(\ell)}{\rho(\ell+1)} \psi(\ell+1) \frac{\Delta\rho(\ell-\tau)}{\phi(\ell-\tau)} \\ &\leq \frac{\Delta\rho(\ell)}{\rho(\ell+1)} \psi(\ell+1) + \frac{\rho(\ell) \Delta(L_1\phi(\ell))}{\phi(\ell-\tau)} - \frac{\rho(\ell)}{\rho^2(\ell+1)} \frac{\psi^2(\ell+1)}{d_1(\ell-\tau)}. \end{aligned} \tag{2.20}$$

Summing (E_s) from ℓ_2 to $\ell-1$ and using (2.19), we obtain

$$-L_2\phi(\ell) \geq -L_2\phi(\ell_2) + \sum_{s=\ell_2}^{\ell-1} F(s) \phi^\beta(s-\tau) \geq \left(\frac{\phi(\ell-\tau)}{D_1(\ell-\tau)} \right)^\beta \sum_{s=\ell_2}^{\ell-1} F(s) D_1^\beta(s-\tau)$$

or

$$-\frac{\Delta(L_1\phi(\ell))}{\phi(\ell-\tau)} \geq \frac{1}{d_2^{1/\beta}(\ell) D_1(\ell-\tau)} \left(\sum_{s=\ell_2}^{\ell-1} F(s) D_1^\beta(\ell-\tau) \right)^{1/\beta} \tag{2.21}$$

for $\ell \geq \ell_3 \geq \ell_2$. Combining (2.20) and (2.21), we have

$$\begin{aligned}\Delta\psi(\ell) \leq &-\frac{\rho(\ell)}{D_1(\ell-\tau)}\left(\frac{1}{d_2(\ell)}\sum_{s=\ell_2}^{\ell-1}F(s)D_1^{\beta}(s-\tau)\right)^{1/\beta}\\ &+\frac{\Delta\rho(\ell)}{\rho(\ell+1)}\psi(\ell+1)-\frac{\rho(\ell)}{\rho^2(\ell+1)}\frac{\psi^2(\ell+1)}{d_1(\ell-\tau)}\\ \leq &-\frac{\rho(\ell)}{D_1(\ell-\tau)}\left(\frac{1}{d_2(\ell)}\sum_{s=\ell_2}^{\ell-1}F(s)D_1^{\beta}(s-\tau)\right)^{1/\beta}\\ &+\frac{d_1(\ell-\tau)}{4}\frac{(\Delta\rho(\ell))^2}{\rho(\ell+1)}.\end{aligned}$$

Summing the last inequality from ℓ_3 to ℓ gives

$$\omega(\ell+1) \leq \omega(\ell_3)-\sum_{s=\ell_3}^{\ell}\left[\frac{\rho(s)}{D_1(s-\tau)}\left(\frac{1}{d_2(s)}\sum_{d=\ell_2}^{s-1}F(j)D_1^{\beta}(j-\tau)\right)^{1/\beta}-\frac{d_1(s-\tau)(\Delta\rho(s))^2}{4\rho(s+1)}\right].$$

Taking the lim sup as $\ell \to \infty$ gives a contradiction to $\omega(\ell) > 0$. The proof is now complete. □

In our next result we obtain sufficient conditions for certain types of solutions to converge to zero.

Theorem 2.10. *Let $\{x(\ell)\}$ be a solution of (E) with the corresponding sequence $\{\phi(\ell)\}$ belonging to the class N_0. If either*

$$\sum_{\ell=\ell_0}^{\infty}F(\ell)=\infty \tag{2.22}$$

or

$$\sum_{\ell=\ell_2}^{\infty}\frac{1}{d_1(\ell)}\sum_{s=\ell}^{\infty}\left(\frac{1}{d_2(s)}\sum_{j=s}^{\infty}F(j)\right)^{1/\beta}=\infty, \tag{2.23}$$

then $\lim_{\ell\to\infty}\phi(\ell)=\lim_{\ell\to\infty}\frac{x(\ell)}{B_1(\ell)}=0$.

Proof. Assume that $\{x(\ell)\}$ is an eventually positive solution of (E). By Corollary 2.3, the corresponding sequence $\{\phi(\ell)\}$ is a positive solution of (E_s), so assume $\phi(\ell) \in N_0$ for $\ell \geq \ell_1$. Since $\phi(\ell)$ is positive and decreasing it has a finite limit, say, $\lim_{\ell\to\infty}\phi(\ell) = \lambda \geq 0$.

Assume that $\lambda > 0$. Summing (E_s) from ℓ_1 to $\ell-1$ and using condition (2.22), we obtain

$$L_2\phi(\ell)=L_2\phi(\ell_1)-\sum_{s=\ell_1}^{\ell-1}F(s)\phi^{\beta}(s-\tau)\leq L_2\phi(\ell_1)-\lambda^{\beta}\sum_{s=\ell_1}^{\ell-1}F(s)\to-\infty \quad \text{as} \quad \ell\to\infty$$

which is a contradiction. Hence, $\lim_{\ell\to\infty}\phi(\ell)=\lim_{\ell\to\infty}\frac{x(\ell)}{B_1(\ell)}=0$.

To show the same conclusion holds in the case where

$$\sum_{\ell=\ell_0}^{\infty} F(s) < \infty,$$

we refer the reader to [21, Lemma 4]. This completes the proof. □

Our next two theorems give conditions to ensure that $N_0 = \emptyset$.

Theorem 2.11. *Assume that*

$$\limsup_{\ell\to\infty} \sum_{s=\ell-\tau}^{\ell-1} F(s) R^{\beta}(\ell-\tau, s-\tau) > 1, \tag{2.24}$$

where

$$R(\ell-\tau, s-\tau) = \sum_{t=s-\tau}^{\ell-\tau} \frac{1}{d_1(t)} \sum_{j=s}^{\ell-\tau} \frac{1}{d_2^{1/\beta}(j)}.$$

Then the class N_0 is empty.

Proof. Assume that $N_0 \neq \emptyset$. Take $\ell_1 \in \mathbb{N}(\ell_0)$ so that $\phi(\ell) \in N_0$ with $\phi(\ell-\tau) > 0$ for $\ell \geq \ell_1$. Using the monotonicity of $L_2\phi(\ell)$ and taking $u > v$ leads to

$$-L_1\phi(v) \geq \sum_{s=v}^{u} \frac{1}{d_2^{1/\beta}(s)} d_2^{1/\beta}(s)\Delta(L_1\phi(s)) \geq d_2^{1/\beta}(u)\Delta(L_1\phi(u)) \sum_{s=v}^{u} \frac{1}{d_2^{1/\beta}(s)}.$$

Summing again from v to u gives

$$\phi(v) \geq d_2^{1/\beta}(u)\Delta(L_1\phi(v)) \sum_{s=v}^{u} \frac{1}{d_1(s)} \sum_{t=s}^{u} \frac{1}{d_2^{1/\beta}(t)} = d_2^{1/\beta}(u)\Delta(L_1\phi(v)) R(u,v) \tag{2.25}$$

Summing (E_s) from $\ell-\tau$ to $\ell-1$ and then using (2.25) with $v = s-\tau$ and $u = \ell-\tau$, we obtain

$$L_2\phi(\ell-\tau) \geq \sum_{s=\ell-\tau}^{\ell-1} F(s)\phi^{\beta}(s-\tau) \geq L_2\phi(\ell-\tau) \sum_{s=\ell-\tau}^{\ell-1} F(s) R^{\beta}(\ell-\tau, s-\tau).$$

Dividing the last inequality by $L_2\phi(\ell-\tau)$ and then taking the lim sup as $\ell \to \infty$ gives a contradiction to (2.24), and completes the proof. □

Theorem 2.12. *Assume that there exists an increasing sequence of integers $\{\xi(\ell)\}$ such that*

$$\xi(\ell) \leq \ell - 1 \quad \text{and} \quad \lim_{\ell\to\infty} \xi(\ell) = \infty. \tag{2.26}$$

If

$$\limsup_{\ell\to\infty} \sum_{s=\ell-\tau}^{\ell} \frac{1}{d_1(s)} \sum_{j=s}^{\xi(\ell)} \left(\frac{1}{d_2(j)} \sum_{i=j}^{\xi(\ell)} F(i) \right)^{1/\beta} > 1, \tag{2.27}$$

then the class N_0 is empty.

Proof. Assume that $\phi(\ell) \in N_0$ with $\ell_1 \in \mathbb{N}(\ell_0)$ such that $\phi(\ell-\tau) > 0$ for $\ell \geq \ell_1$. Summing (E_s) from i to $\xi(\ell)$ yields

$$L_2\phi(i) \geq \sum_{j=i}^{\xi\{\ell\}} F(j)\phi^{\beta}(j-\tau).$$

Since $\{\phi(\ell)\}$ is decreasing,

$$\Delta(L_1\phi(i)) \geq \frac{\phi(\ell-\tau)}{d_2^{1/\beta}(i)} \left(\sum_{j=i}^{\xi(\ell)} F(j)\right)^{1/\beta}.$$

Summing this inequality from i to $\xi(\ell)$,

$$-\Delta\phi(i) \geq \frac{\phi(\ell-\tau)}{d_1(\ell)} \sum_{s=i}^{\xi(\ell)} \left(\frac{1}{d_2(s)} \sum_{j=s}^{\xi(\ell)} F(j)\right)^{1/\beta}.$$

Summing once more from i to ℓ, we obtain

$$\phi(i) \geq \phi(\ell-\tau) \sum_{k=i}^{\ell} \frac{1}{d_1(k)} \sum_{s=k}^{\xi(\ell)} \left(\frac{1}{d_2(s)} \sum_{j=s}^{\xi(\ell)} F(j)\right)^{1/\beta}.$$

Letting $i = \ell - \tau$ in the above inequality and then taking the lim sup as $\ell \to \infty$, gives a contradiction to (2.27). The proof is now complete. □

3. **Main Oscillation Results**

By combining Theorem 2.10 with either Theorem 2.6, Theorem 2.7, Theorem 2.8, or Theorem 2.9, we obtain the following theorem.

Theorem 3.1. *If* (2.8) *(or* (2.12) *or* (2.16) *or* (2.18)*) and either* (2.22) *or* (2.23) *hold, then every solution* $x(\ell)$ *of* (E) *is either oscillatory or satisfies* $\lim_{\ell\to\infty} \frac{x(\ell)}{B_1(\ell)} = 0$.

Proof. Let $\{x(\ell)\}$ be an eventually positive solution of (E), say $x(\ell) > 0$ and $x(\ell-\tau) > 0$ for $\ell \geq \ell_1 \in \mathbb{N}(\ell_0)$. By Corollary 2.3, the corresponding sequence $\{\phi(\ell)\} = \{\frac{x(\ell)}{B_1(\ell)}\}$ is a positive solution of (E_s) and so $\phi(\ell) \in N_0 \cup N_1 \cup N_2$ for $\ell \geq \ell_1$.

From Theorem 2.6 (or Theorem 2.7 or Theorem 2.8 or Theorem 2.9), we see that class $N_1 \cup N_2$ is empty and so $\phi(\ell) \in N_0$. But by Theorem 2.10, we see that $\lim_{\ell\to\infty} \phi(\ell) = 0$ so that $\lim_{\ell\to\infty} \dfrac{x(\ell)}{B_1(\ell)} = 0$, which proves the theorem. □

Combining Theorem 2.6 or Theorem 2.7 or Theorem 2.8 or Theorem 2.9 with either Theorem 2.11 or Theorem 2.12 yields the following oscillation results.

Theorem 3.2. *In addition to condition* (2.24)*, assume that the conditions of Theorem 2.6 (or Theorem 2.7 or Theorem 2.8 or Theorem 2.9) hold. Then every solution of* (E) *is oscillatory.*

Proof. Assume that $\{x(\ell)\}$ is an eventually positive solution of (E), say $x(\ell) > 0$ and $x(\ell - \tau) > 0$ for all $\ell \geq \ell_1 \in \mathbb{N}(\ell_0)$. By Corollary 2.3, the corresponding sequence $\{\phi(\ell)\} = \{\frac{x(\ell)}{B_1(\ell)}\}$ is a positive solutions of (E_s), and so $\phi(\ell) \in N_0 \cup N_1 \cup N_2$ for all $\ell \geq \ell_1$.

To prove the theorem, we want to show that the classes N_0, N_1, and N_2 are empty. From Theorem 2.6 (or Theorem 2.7 or Theorem 2.8 or Theorem 2.9), we see that the class $N_1 \cup N_2$ is empty. Also, by Theorem 2.11, N_0 is empty. Hence, (E_s) is oscillatory, and so the oscillation preserving transformation implies that (E) is oscillatory. This completes the proof. □

Theorem 3.3. *In addition to* (2.26) *and* (2.27) *assume that the conditions of Theorem 2.6 (or Theorem 2.7 or Theorem 2.8 or Theorem 2.9) hold. Then every solution of* (E) *is oscillatory.*

Proof. The proof is similar to that of Theorem 3.2 and so the details are omitted. □

4. Examples

In this section, we present examples to illustrate our main results.

Example 4.1. Consider the third-order noncanonical delay difference equation

$$\Delta(\ell(\ell+1)\Delta((\ell+1)(\ell+2)\Delta x(\ell))) + a\ell x(\ell-1) = 0, \quad \ell \geq 2, \tag{4.1}$$

where $a > 0$ is a constant. Here we have $b_1(\ell) = (\ell+1)(\ell+2)$, $b_2(\ell) = \ell(\ell+1)$, $\beta = 1$, $f(\ell) = a\ell$, $\tau = 1$, and $\ell_0 = 2$, so $B_1(\ell) = \frac{1}{\ell+1}$, $d_1(\ell) = 1$, $d_2(\ell) = \ell(\ell+1)(\ell+2)$, $D_1(\ell) \approx \ell$, $D_2(\ell) = \frac{1}{2\ell(\ell+1)}$, and $F(\ell) = a$.

By simple computations, equation (4.1) can be transformed into the semi-canonical equation

$$\Delta(\ell(\ell+1)(\ell+2)\Delta^2\phi(\ell)) + a\phi(\ell-1) = 0, \quad \ell \geq 2.$$

Condition (2.6) becomes

$$\sum_{\ell=2}^{\infty} \frac{1}{\ell(\ell+1)(\ell+2)} \sum_{s=2}^{\ell-1} a(s-1) \approx a \sum_{\ell=2}^{\infty} \frac{1}{\ell} = \infty,$$

so it holds. Condition (2.16) becomes

$$\limsup_{\ell\to\infty} \frac{1}{2\ell(\ell+1)} \sum_{s=2}^{\ell-1} a(s-1) = \frac{a}{4} > 1,$$

so it is satisfied if $a > 4$. Condition (2.22) is clearly satisfied. Therefore by Theorem 3.1, any solution of (4.1) in either oscillatory or satisfies $\lim_{\ell\to\infty}(\ell+1)x(\ell) = 0$ for $a > 4$.

Example 4.2. Consider the third-order non-canonical delay difference equation

$$\Delta(2^\ell\Delta(2^\ell\Delta x(\ell)) + a4^\ell x(\ell-2) = 0, \quad \ell \geq 3, \tag{4.2}$$

where $a > 0$ is a constant. Here, $b_1(\ell) = 2^\ell = b_2(\ell)$, $\beta = 1$, $f(\ell) = a4^\ell$, $\tau = 2$, and $\ell_0 = 3$, so $B_1(\ell) = \frac{2}{2^\ell}$, $d_1(\ell) = \frac{2}{2^\ell}$, $d_2(\ell) = 4^\ell$, and $F(\ell) = 8a2^\ell$. The transformed equation becomes

$$\Delta(4^\ell\Delta\left(\frac{1}{2^\ell}\Delta\phi(\ell)\right)) + 4a2^\ell\phi(\ell-2) = 0,\ \ell \geq 3,$$

and $D_1(\ell) \approx 2^\ell$ with simplified $F(\ell) = 4a2^\ell$. Condition (2.12) becomes

$$\liminf_{\ell\to\infty} \sum_{s=\ell-2}^{\ell-1}\left(\frac{1}{4^s}\sum_{j=3}^{s-1} a4^j\right) = \frac{2a}{3} > \frac{8}{27},$$

so it is satisfied if $a > \frac{4}{9}$. By taking $\{\xi(\ell)\} = \{\ell-1\}$, we see that (2.26) holds; (2.27) becomes

$$\limsup_{\ell\to\infty} \sum_{s=\ell-2}^{\ell-1} 2^s \sum_{j=s}^{\ell-1}\frac{1}{4^j}\sum_{i=j}^{\ell-1} 4a2^i = 18a > 1,$$

which means it holds if $a > \frac{1}{18}$. Therefore, by Theorem 3.3, equation (4.2) is oscillatory for $a > \frac{4}{9}$.

Example 4.3. Consider the third-order non-canonical delay difference equation

$$\Delta(\ell(\ell+1)\Delta((\ell+1)(\ell+2)\Delta x(\ell))) + a\ell 2^\ell x(\ell-1) = 0, \quad \ell \geq 2, \tag{4.3}$$

where again $a > 0$ is a constant. Here we have $b_1(\ell) = (\ell+1)(\ell+2)$, $b_2(\ell) = \ell(\ell+1)$, $\beta = 1$, $f(\ell) = a\ell 2^\ell$, $\tau = 1$, and $\ell_0 = 2$, so $B_1(\ell) = \frac{1}{\ell+1}$, $d_1(\ell) = 1$, $d_2(\ell) = \ell(\ell+1)(\ell+2)$, $D_1(\ell) \approx \ell$, $D_2(\ell) = \frac{1}{2\ell(\ell+1)}$, and $F(\ell) = a2^\ell$. The transformed equation becomes

$$\Delta(\ell(\ell+1)(\ell+2)\Delta^2\varphi(\ell)) + a2^\ell\varphi(\ell-1) = 0, \quad \ell \geq 2.$$

Condition (2.6) is clearly satisfied for $a > 0$, and by choosing $\rho(\ell) = 1$, we see that (2.18) is also satisfied for $a > 0$. Moreover,

$$R(\ell-1, s-1) = \sum_{t=s-1}^{\ell-1}\left(\sum_{j=t}^{\ell-1}\frac{1}{j(j+1)(j+2)}\right) = \frac{(\ell-s+1)(\ell-s+2)}{2\ell(\ell+1)(s-1)}.$$

Now condition (2.24) takes the form

$$\limsup_{\ell\to\infty} \sum_{s=\ell-2}^{\ell-1} a\ 2^{s-1}\frac{(\ell-s+1)(\ell-s+2)}{\ell(\ell+1)(s-1)} = \lim_{\ell\to\infty}\frac{3a2^\ell(2\ell-5)}{2\ell(\ell+1)(\ell-2)(\ell-3)} = \infty > 1.$$

Therefore, Theorem 2.9 holds, and thus by Theorem 3.2, every solution of (4.3) is oscillatory.

Example 4.4. Consider the third-order half-linear delay difference equation

$$\Delta(8^{\ell}(\Delta(2^{\ell}\Delta x(\ell)))^3) + a2^{6\ell}x^3(\ell-1) = 0, \quad \ell \geq 2, \tag{4.4}$$

where $a > 0$ is a constant.

Here we have $b_1(\ell) = 2^{\ell}$, $b_2(\ell) = 8^{\ell}$, $\beta = 3$, $f(\ell) = a2^{6\ell}$, $\tau = 1$, $\ell_0 = 2$, $B_1(\ell) = \frac{2}{2^{\ell}}$, $d_1(\ell) = \frac{2}{2^{\ell}}$, $d_2(\ell) = \frac{2^{6\ell}}{8}$, and $F(\ell) = 64a2^{3\ell}$. The transformed equation is

$$\Delta(2^{6\ell}(\Delta(\frac{1}{2^{\ell}}\Delta\phi(\ell))^3) + 64a2^{3\ell}\phi(\ell-1) = 0, \quad \ell \geq 2$$

which is in semi-canonical form. Further $D_1(\ell) = 2^{\ell}$. Now condition (2.12) becomes

$$\lim_{\ell\to\infty}\inf \sum_{s=\ell-1}^{\ell-1}\left(\frac{1}{2^{6s}}\sum_{j=2}^{s-1}8a2^{6j}\right)^{\frac{1}{3}} = \left(\frac{8a}{63}\right)^{\frac{1}{3}} > \frac{1}{4},$$

which holds if $a > 0.123047$. In addition,

$$R(\ell-1, s-1) = \sum_{t=s-1}^{\ell-1} 2^t \sum_{j=t}^{\ell-1}\frac{1}{4^j} = 2\left(\frac{8}{3}\frac{1}{2^s} - \frac{2}{2^{\ell}} + \frac{1}{3}\frac{2^s}{4^{\ell}}\right)$$

and using this, condition (2.24) becomes

$$\limsup_{\ell\to\infty}\sum_{s=\ell-1}^{\ell-1} 512a2^{3s}\left(\frac{8}{3}\frac{1}{2^s} - \frac{2}{2^{\ell}} + \frac{1}{3}\frac{2^s}{4^{\ell}}\right)^3 = 64a(\frac{7}{2})^3 > 1,$$

so it holds for $a > .0003645$. Therefore, Theorem 2.7 holds, so by Theorem 3.2, every solution of (4.4) is oscillatory if $a > 0.123047$.

5. **Conclusion**

In this paper, we have obtained some new oscillation criteria for equation (E) by transforming it into a semi-canonical type equation, which we did without assuming any extra conditions. The results obtained are new and complement those in [4, 5, 12, 16, 18, 21, 24, 25, 27]. Examples are provided to illustrate the main results.

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