

EXISTENCE RESULTS FOR NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING THE KATUGAMPOLA DERIVATIVE WITH VARIABLE ORDER

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ABSTRACT. The authors introduce a fractional derivative with a variable order based on the Katugampola fractional derivative and integral. They show how to handle an implicit Dirichlet boundary value problem by applying a fixed point theorem for the sum of a contraction and compact maps due to Krasnosel'skii. They show that solutions to the problem exist and are unique. An example illustrates the results.

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1. INTRODUCTION

The transition from constant order fractional calculus to variable order fractional calculus is a significant conceptual shift. Instead of having a fixed order parameter \mathcal{K} as in the constant order case, the variable order approach introduces a dynamic feature by replacing \mathcal{K} with a time-varying function $\mathcal{K}(t)$. While this may seem to be a subtle change, there are significant implications of doing this. The variable order operator enables modeling a broader spectrum of physical phenomena.

The origin of variable order fractional calculus can be traced to the work of Samko and Ross [13] in 1993. They explored situations in which the differentiation or integration order varied with time; we will designate this by $\mathcal{K}(t)$. This initiated a reexamination and a revision of the classical fractional definitions, such as those of

Riemann-Liouville and Caputo type, to accommodate the variability in the variable order calculus.

Through the variable order calculus, researchers obtained a versatile tool for analyzing complex systems. It provides the flexibility needed to model real-world processes characterized by evolving dynamics and non-linearities. This leads to a deeper understanding of the interplay between variables and leads to more accurate analyses of dynamical behavior. In view of this, several efforts have been devoted to variable order fractional operators and their applications (see, for example, Almeida and Samko [1] and Lorenzo and Hartley [7]). Samko and Ross' [13] work on variable order fractional calculus has become a fundamental tool in mechanics and the study of viscous flows. Numerous physical phenomena demonstrate fractional-order dynamics that may evolve over time or space. Coimbra [2] did an in-depth study of fractional differential equations of variable order type. The paper by Diaz and Coimbra [3] contains an analysis of the dynamics of a nonlinear variable viscoelastic oscillator and presents two controllers designed specifically for variable order differential equations. Pedro *et al.* [11] investigated the drag force acting on particles in oscillatory flows of viscous fluids. All these papers indicate the versatility of the variable order fractional calculus in modeling and understanding complex systems.

Boundary value problems for fractional differential equations have been a very active area of research in recent years. In this paper, we present a new fractional integral that extends the fractional integrals of Riemann-Liouville, Hadamard, Erdlyi-Kober, Katugampola, Weyl, and Liouville type and in some sense unifies them. The Katugampola derivative and integral [4, 5] also provide a broader perspective on the Hadamard and Riemann-Liouville derivatives and integrals.

In a very nice study [18], Zhang presented an insightful introduction to variable order fractional derivatives and how they are used in initial value problems for fractional differential equations. Sun *et al.* gave an in-depth survey of variable order fractional derivatives and physical problems associated with them.

In [9], Henderson and Maazouz studied the existence of solutions to the Katugampola fractional boundary value problems

$$\begin{cases} {}^\rho \mathfrak{Y}^\mathcal{K} \vartheta(\iota) + \mathcal{Q}(\iota, \vartheta(\iota)) = 0, & \iota \in \mathcal{I} = [a, b], \quad 1 < \mathcal{K} < 2, \quad \rho > 0, \\ \vartheta(a) = \vartheta(b) = 0, \end{cases}$$

and

$$(1.1) \quad \begin{cases} {}^\rho \mathfrak{Y}^\mathcal{K} \vartheta(\iota) + \widehat{\mathcal{Q}}(\iota, \vartheta(\iota), {}^\rho \mathfrak{Y}^\mathcal{K} \vartheta(\iota)) = 0, & \iota \in \mathcal{I} = [a, b], \quad 1 < \mathcal{K} < 2, \quad \rho > 0, \\ \vartheta(a) = \vartheta(b) = 0, \end{cases}$$

where ${}^\rho \mathfrak{Y}^\mathcal{K}$ is the Katugampola fractional derivative of order \mathcal{K} , $\mathcal{Q} : \mathcal{I} \times \mathbb{R} \rightarrow \mathbb{R}$, and $\widehat{\mathcal{Q}} : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, and $0 \leq a < b < \infty$.

In this paper we will examine the existence and uniqueness of solutions to the implicit nonlinear boundary value problem with a variable-order Katugampola fractional derivative

$$(1.2) \quad \begin{cases} {}^\rho \mathfrak{J}^{\mathcal{K}(\iota)} \vartheta(\iota) + \mathcal{Q}(\iota, \vartheta(\iota)) = 0, \quad \iota \in \mathcal{I} = [a, b], \quad 1 < \mathcal{K}(\iota) < 2, \quad \rho > 0, \\ \vartheta(a) = \vartheta(b) = 0, \end{cases}$$

where ${}^\rho \mathfrak{J}^{\mathcal{K}(\iota)}$ is the Katugampola fractional derivative of variable order $\mathcal{K}(\iota)$, $\mathcal{Q} : \mathcal{I} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and $0 \leq a < b < \infty$. The Katugampola fractional derivative of variable order was not mentioned in either of the studies [16, 18], but problems involving the Katugampola fractional derivative have received considerable attention in recent years; see for example, the papers [4, 5, 8, 9, 10, 17] and the references therein.

Our main tool will be a fixed point theorem due to Krasnosel'skii for the sum of a contraction and a compact map.

Our paper is organized as follows: In Section 2, we present definitions, notations, and lemmas needed to prove our main results that are in turn contained in Section 3. We conclude the paper with an example to illustrate our results.

2. PRELIMINARIES

Let \mathfrak{E} be a Banach space and let Ω be a subset of \mathfrak{E} . We define $C(\Omega, \mathfrak{E})$ to be the Banach space of continuous functions $\vartheta : \Omega \rightarrow \mathfrak{E}$ equipped with the usual supremum norm

$$\|\vartheta\|_\infty = \sup\{|\vartheta(\iota)| : \iota \in \Omega\}.$$

In what follows, $\Gamma(\cdot)$ is the usual Gamma function and $[i]$ denotes the greatest integer less than or equal to i .

Definition 2.1. Let $-\infty < a < \iota < +\infty$ and $\mathcal{K} : [a, \iota] \rightarrow (0, +\infty)$. The generalized left-hand fractional integral of variable order of the function h is defined by

$${}^\rho I^{\mathcal{K}(\iota)} h(\iota) = \frac{\rho^{1-\mathcal{K}(\iota)}}{\Gamma(\mathcal{K}(\iota))} \int_a^\iota (\iota^\rho - s^\rho)^{\mathcal{K}(\iota)-1} s^{\rho-1} h(s) ds, \quad \iota > a$$

provided the integral exists.

Definition 2.2. The generalized left-hand fractional derivative of variable order of the function h , corresponding to the generalized fractional integral, is defined by

$${}^\rho \mathfrak{J}^{\mathcal{K}(\iota)} h(\iota) = \frac{\rho^{\mathcal{K}(\iota)-n+1}}{\Gamma(n - \mathcal{K}(\iota))} \left(\iota^{1-\rho} \frac{d}{d\iota} \right)^n \int_a^\iota (\iota^\rho - s^\rho)^{n-\mathcal{K}(\iota)-1} s^{\rho-1} h(s) ds, \quad \iota > a,$$

where $n = [\mathcal{K}(\iota)] + 1$.

The interested reader might wish to compare the definitions we give here to those for fixed order derivatives of Katugampola type found in [4, 5].

In the case where $\mathcal{K}(t)$ is a constant, we have the following two lemmas.

Lemma 2.3. ([9, Lemma 2.1 corrected]) *Let $\mathcal{K} > 0$ and $\rho > 0$. Then the fractional differential equation*

$${}^{\rho}\mathfrak{J}^{\mathcal{K}}\sigma(\lambda) = 0$$

has the solution

$$\sigma(\lambda) = \sum_{i=0}^{m-1} a_i \left(\frac{\lambda^{\rho} - a^{\rho}}{\rho} \right)^{\mathcal{K}-i-1}, \quad a_i \in \mathbb{R}, \quad i = 0, 1, 2, \dots, m-1,$$

where $m = [\mathcal{K}] + 1$.

Lemma 2.4. ([9, Lemma 2.2 corrected]) *Let $\mathcal{K} > 0$ and $\rho > 0$. Then*

$${}^{\rho}\mathcal{I}^{\mathcal{K}} ({}^{\rho}\mathfrak{J}^{\mathcal{K}}\sigma(\lambda)) = \sigma(\lambda) + \sum_{i=0}^{m-1} a_i \left(\frac{\lambda^{\rho} - a^{\rho}}{\rho} \right)^{\mathcal{K}-i-1},$$

where $a_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, m-1$, and $m = [\mathcal{K}] + 1$.

In addition to the above two lemmas, we will need the following fixed point theorem of Krasnosel'skii to prove our main result.

Theorem K. ([14, Theorem 4.4.1]) *Let Ω be a non-empty, closed, bounded, and convex subset of a real Banach space E and let F_1 and F_2 be operators on Ω satisfying:*

- (a) $F_1(\Omega) + F_2(\Omega) \subseteq \Omega$;
- (b) F_1 is a strict contraction, that is, there exists a constant $k \in [0, 1)$ such that

$$\|F_1(x) - F_1(y)\| \leq k\|x - y\|$$

for all $x, y \in \Omega$;

- (c) F_2 is continuous and $F_2(\Omega)$ is a relatively compact subset of E .

Then, $F_1(x) + F_2(x) = x$ admits at least one solution in Ω .

3. MAIN RESULTS

This section contains our main existence and uniqueness results. We begin with the following concepts. First note that $[\mathcal{K}(\lambda)] = 1$ so that in what follows, for our problem we have $n = [\mathcal{K}(\lambda)] + 1 = 2$.

Definition 3.1. [18] Let \mathfrak{S} be a subset of \mathbb{R} .

- (i) \mathfrak{S} is called a generalized interval if it is either a standard interval, a point, or \emptyset .

- (ii) If \mathfrak{S} is a generalized interval, then the finite set \mathcal{P} consisting of generalized intervals $\{J_1, J_2, \dots, J_k\} = \{[a, \lambda_1], (\lambda_1, \lambda_2], (\lambda_2, \lambda_3], \dots, (\lambda_{k-1}, b]\}$ contained in \mathfrak{S} is called a partition of \mathfrak{S} provided that each $\lambda \in \mathfrak{S}$ lies in exactly one of the generalized intervals J_i in the finite set \mathcal{P} .
- (iii) We say that the function $\aleph : \lambda \rightarrow \mathbb{R}$ is piece-wise constant with respect to the partition \mathcal{P} of \mathfrak{S} , if for each $J_i \in \mathcal{P}$, \aleph is constant on J_i , $i=1,2,\dots, k$.

As was done in [18], we assume that the fractional order $\mathcal{K} : [a, b] \rightarrow (1, 2)$ is a piecewise constant function with respect to the partition \mathcal{P} so that

$$\mathcal{K}(\lambda) = \sum_{i=1}^k \mathcal{K}_i \chi_i(\lambda), \quad \text{for } \lambda \in [a, b],$$

where $1 < \mathcal{K}_i < 2$ for each $i = 1, 2, \dots, k$, the constants \mathcal{K}_i are defined on the intervals J_i of the partition \mathcal{P} for $i = 1, 2, \dots, k$, and χ_i denotes the indicator (characteristic) function for the interval $[\lambda_{i-1}, \lambda_i]$ as given by

$$\chi_i(\lambda) = \begin{cases} 1, & \text{if } \lambda \in [\lambda_{i-1}, \lambda_i], \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda_0 = a$ and $\lambda_k = b$. Thus, according to the definition, since $n = 2$,

$$\begin{aligned} {}^\rho \mathfrak{J}^{\mathcal{K}(\lambda)} h(\lambda) &= \frac{\rho^{\mathcal{K}(\lambda)-2+1}}{\Gamma(2 - \mathcal{K}(\lambda))} \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_a^\lambda (\lambda^\rho - s^\rho)^{2-\mathcal{K}(\lambda)-1} s^{\rho-1} h(s) ds, \\ &= \frac{\rho^{\mathcal{K}(\lambda)-1}}{\Gamma(2 - \mathcal{K}(\lambda))} \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_a^\lambda (\lambda^\rho - s^\rho)^{1-\sum_{i=1}^n \mathcal{K}_i \chi_i(\lambda)} s^{\rho-1} h(s) ds, \\ &= \frac{\rho^{\mathcal{K}(\lambda)-1}}{\Gamma(2 - \mathcal{K}(\lambda))} \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_a^{\lambda_1} (\lambda^\rho - s^\rho)^{1-\mathcal{K}_1} s^{\rho-1} h(s) ds \\ &\quad + \frac{\rho^{\mathcal{K}(\lambda)-1}}{\Gamma(2 - \mathcal{K}(\lambda))} \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_{\lambda_1}^{\lambda_2} (\lambda^\rho - s^\rho)^{1-\mathcal{K}_2} s^{\rho-1} h(s) ds \\ &\quad + \dots + \frac{\rho^{\mathcal{K}(\lambda)-1}}{\Gamma(2 - \mathcal{K}(\lambda))} \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_{\lambda_n}^\lambda (\lambda^\rho - s^\rho)^{1-\mathcal{K}_n} s^{\rho-1} h(s) ds, \\ &= \frac{\rho^{\mathcal{K}(\lambda)-1}}{\Gamma(2 - \mathcal{K}(\lambda))} \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \left[\sum_{i=1}^n \int_{\lambda_{i-1}}^{\lambda_i} (\lambda^\rho - s^\rho)^{1-\mathcal{K}_i} s^{\rho-1} h(s) ds \right. \\ &\quad \left. + \int_{\lambda_n}^\lambda (\lambda^\rho - s^\rho)^{1-\mathcal{K}_n} s^{\rho-1} h(s) ds \right], \\ &= \frac{\rho^{\mathcal{K}(\lambda)-1}}{\Gamma(2 - \mathcal{K}(\lambda))} \left[\left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \sum_{i=1}^n \int_{\lambda_{i-1}}^{\lambda_i} (\lambda^\rho - s^\rho)^{1-\mathcal{K}_i} s^{\rho-1} h(s) ds \right. \\ &\quad \left. + \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_{\lambda_n}^t (\lambda^\rho - s^\rho)^{1-\mathcal{K}_n} s^{\rho-1} h(s) ds \right]. \end{aligned} \tag{3.1}$$

Therefore, in the interval $[a, \lambda_1]$, the expression (3.1) becomes

$${}^\rho \mathfrak{J}_a^{\mathcal{K}_1} h(\lambda) = \frac{\rho^{\mathcal{K}_1-1}}{\Gamma(2-\mathcal{K}_1)} \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_a^{\lambda_1} (\lambda^\rho - s^\rho)^{1-\mathcal{K}_1} s^{\rho-1} h(s) ds,$$

and in $(\lambda_1, \lambda_2]$, it is

$$\begin{aligned} {}^\rho \mathfrak{J}_a^{\mathcal{K}_2} h(\lambda) &= \frac{\rho^{\mathcal{K}_2-1}}{\Gamma(2-\mathcal{K}_2)} \left[\left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_a^{\lambda_1} (\lambda^\rho - s^\rho)^{1-\mathcal{K}_2} s^{\rho-1} h(s) ds \right. \\ &\quad \left. + \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_{\lambda_1}^{\lambda} (\lambda^\rho - s^\rho)^{1-\mathcal{K}_2} s^{\rho-1} h(s) ds \right]. \end{aligned}$$

In general, for the interval $(\lambda_{i-1}, \lambda_i]$, for $i \in \{1, \dots, k\}$, the expression (3.1) becomes

$$\begin{aligned} {}^\rho \mathfrak{J}_a^{\mathcal{K}_i} h(\lambda) &= \frac{\rho^{\mathcal{K}_i-1}}{\Gamma(2-\mathcal{K}_i)} \left[\left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \sum_{i=1}^{n-1} \int_a^{\lambda_i} (\lambda^\rho - s^\rho)^{1-\mathcal{K}_i} s^{\rho-1} h(s) ds \right. \\ &\quad \left. + \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_{\lambda_{i-1}}^{\lambda} (\lambda^\rho - s^\rho)^{1-\mathcal{K}_i} s^{\rho-1} h(s) ds \right]. \end{aligned}$$

For each $i \in \{1, 2, \dots, k\}$, let $\mathfrak{H}_i = C^1([\lambda_{i-1}, \lambda_i], \mathbb{R})$ denote a Banach space with the norm

$$\|\vartheta\|_{\mathfrak{H}_i} = \sup_{\lambda \in [\lambda_{i-1}, \lambda_i]} \{|\vartheta(\lambda)|\}.$$

Now for each $i = 1, 2, \dots, k$, consider the auxiliary boundary value problem for Katugampola fractional derivatives of constant order

$$(3.2) \quad \begin{cases} {}^\rho \mathfrak{J}_a^{\mathcal{K}_i} \hat{\vartheta}(\lambda) = \frac{\rho^{\mathcal{K}_i-1}}{\Gamma(2-\mathcal{K}_i)} \left(\lambda^{1-\rho} \frac{d}{d\lambda} \right)^2 \int_{\lambda_{i-1}}^{\lambda} (\lambda^\rho - s^\rho)^{1-\mathcal{K}_i} s^{\rho-1} \hat{\vartheta}(s) ds \\ \quad = -\mathcal{Q}(\lambda, \hat{\vartheta}(\lambda)), \quad \lambda \in [\lambda_{i-1}, \lambda_i], \\ \hat{\vartheta}(\lambda_{i-1}) = \hat{\vartheta}(\lambda_i) = 0. \end{cases}$$

Definition 3.2. We say that the problem (1.2) has a solution ϑ , if there exist functions $\vartheta_i \in \mathfrak{H}_i$ for each $i = 1, 2, \dots, k$ satisfying (3.2).

Definition 3.3. We will say that problem (3.2) has a unique solution provided that the functions ϑ_i are unique for each $i = 1, 2, \dots, k$.

We next give an integral representation for a solution of our problem.

Lemma 3.4. *The function $\hat{\vartheta}$ is a solution of (3.2) if and only if it satisfies the integral equation*

$$(3.3) \quad \hat{\vartheta}(\lambda) = -{}^\rho I_{\lambda_{i-1}^+}^{\mathcal{K}_i} \mathcal{Q}(\lambda, \hat{\vartheta}(\lambda)) + \left(\frac{\lambda^\rho - \lambda_{i-1}^\rho}{\lambda_i^\rho - \lambda_{i-1}^\rho} \right)^{\mathcal{K}_i-1} \times {}^\rho I_{\lambda_{i-1}^+}^{\mathcal{K}_i} \mathcal{Q}(\lambda_i, \hat{\vartheta}(\lambda_i)), \quad \lambda_{i-1} < \lambda \leq \lambda_i,$$

for each $i = 1, 2, \dots, k$.

Proof. Assume $\hat{\vartheta}_i$ satisfies (3.3). Then for $\lambda_{i-1} < \lambda \leq \lambda_i$,

$${}^{\rho, \mathcal{K}_i} I_{\lambda_{i-1}^+}^{\rho} \mathfrak{Y}_{\lambda_{i-1}^+}^{\mathcal{K}_i} \hat{\vartheta}_i(\lambda) = -{}^{\rho} I_{\lambda_{i-1}^+}^{\mathcal{K}_i} \mathcal{Q}(\lambda, \hat{\vartheta}_i(\lambda)),$$

so that

$$\hat{\vartheta}_i(\lambda) = -{}^{\rho} I_{\lambda_{i-1}^+}^{\mathcal{K}_i} \mathcal{Q}(\lambda, \hat{\vartheta}_i(\lambda)) + \sum_{j=0}^{n-1} a_j \left(\frac{\lambda^{\rho} - \lambda_{i-1}^{\rho}}{\rho} \right)^{\mathcal{K}_i - j - 1}$$

by Lemma 2.4. From the boundary conditions $\hat{\vartheta}(\lambda_i) = \hat{\vartheta}(\lambda_{i-1}) = 0$, we obtain

$$a_1 = 0 \quad \text{and} \quad a_0 = -{}^{\rho} I_{\lambda_{i-1}^+}^{\mathcal{K}_i} \mathcal{Q}(\lambda, \hat{\vartheta}(\lambda)) \div \left(\frac{\lambda^{\rho} - \lambda_{i-1}^{\rho}}{\rho} \right)^{1 - \mathcal{K}_i},$$

so that

$$\hat{\vartheta}(\lambda) = -{}^{\rho} I_{\lambda_{i-1}^+}^{\mathcal{K}_i} \mathcal{Q}(\lambda, \hat{\vartheta}(\lambda)) + \left(\frac{\lambda^{\rho} - \lambda_{i-1}^{\rho}}{\lambda_i^{\rho} - \lambda_{i-1}^{\rho}} \right)^{\mathcal{K}_i - 1} \times {}^{\rho} I_{\lambda_{i-1}^+}^{\mathcal{K}_i} \mathcal{Q}(\lambda_i, \hat{\vartheta}(\lambda_i)), \quad \lambda_{i-1} < \lambda \leq \lambda_i.$$

This can be written as

$$\begin{aligned} \hat{\vartheta}(\lambda) &= \frac{\rho^{1 - \mathcal{K}_i}}{\Gamma(\mathcal{K}_i)} \left[\left(\frac{\lambda^{\rho} - \lambda_{i-1}^{\rho}}{\lambda_i^{\rho} - \lambda_{i-1}^{\rho}} \right)^{\mathcal{K}_i - 1} \int_{\lambda_{i-1}}^{\lambda} (\lambda_i^{\rho} - s^{\rho})^{\mathcal{K}_i - 1} s^{\rho - 1} \mathcal{Q}(s, \hat{\vartheta}(s)) ds \right. \\ &\quad \left. - \int_{\lambda_{i-1}}^{\lambda} (\lambda^{\rho} - s^{\rho})^{\mathcal{K}_i - 1} s^{\rho - 1} \mathcal{Q}(s, \hat{\vartheta}(s)) ds \right. \\ &\quad \left. + \left(\frac{\lambda^{\rho} - \lambda_{i-1}^{\rho}}{\lambda_i^{\rho} - \lambda_{i-1}^{\rho}} \right)^{\mathcal{K}_i - 1} \int_{\lambda_{i-1}}^{\lambda_i} (\lambda_i^{\rho} - s^{\rho})^{\mathcal{K}_i - 1} s^{\rho - 1} \mathcal{Q}(s, \hat{\vartheta}(s)) ds \right] \\ &= \frac{\rho^{1 - \mathcal{K}_i}}{\Gamma(\mathcal{K}_i)} \left[\left(\frac{\lambda^{\rho} - \lambda_{i-1}^{\rho}}{\lambda_i^{\rho} - \lambda_{i-1}^{\rho}} \right)^{\mathcal{K}_i - 1} \int_{\lambda_{i-1}}^{\lambda_i} (\lambda_i^{\rho} - s^{\rho})^{\mathcal{K}_i - 1} - (\lambda_{i-1}^{\rho} - s^{\rho})^{\mathcal{K}_i - 1} s^{\rho - 1} \mathcal{Q}(s, \hat{\vartheta}(s)) ds \right. \\ &\quad \left. + \left(\frac{\lambda^{\rho} - \lambda_{i-1}^{\rho}}{\lambda_i^{\rho} - \lambda_{i-1}^{\rho}} \right)^{\mathcal{K}_i - 1} \int_{\lambda_{i-1}}^{\lambda_i} (\lambda_i^{\rho} - s^{\rho})^{\mathcal{K}_i - 1} s^{\rho - 1} \mathcal{Q}(s, \hat{\vartheta}(s)) ds \right] \\ &= \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda_{i-1}, s) \mathcal{Q}(s, \hat{\vartheta}(s)) ds, \end{aligned}$$

where $G_i(\lambda_{i-1}, s) : [a, b] \times [a, b] \rightarrow \mathbb{R}$ is the Green's function defined by

$$(3.4) \quad G_i(\lambda_{i-1}, s) = \frac{\rho^{1 - \mathcal{K}_i}}{\Gamma(\mathcal{K}_i)} \begin{cases} \left[\left(\frac{\lambda^{\rho} - \lambda_i^{\rho}}{\lambda_{i-1}^{\rho} - \lambda_i^{\rho}} \right)^{\mathcal{K}_i - 1} (\lambda_i^{\rho} - s^{\rho})^{\mathcal{K}_i - 1} - (\lambda_{i-1}^{\rho} - s^{\rho})^{\mathcal{K}_i - 1} \right] s^{\rho - 1}, \\ \quad \text{for } \lambda_{i-1} \leq s \leq \lambda \leq \lambda_i, \\ \left(\frac{\lambda^{\rho} - \lambda_{i-1}^{\rho}}{\lambda_i^{\rho} - \lambda_{i-1}^{\rho}} \right)^{\mathcal{K}_i - 1} (\lambda_i^{\rho} - s^{\rho})^{\mathcal{K}_i - 1} s^{\rho - 1}, \\ \quad \text{for } \lambda_{i-1} \leq \lambda \leq s \leq \lambda_i, \end{cases}$$

□

Lemma 3.5. For the function $G_i(\lambda_{i-1}, s)$ defined by (3.4) with $1 < \mathcal{K}_i(\lambda) < 2$ and $\rho > 0$, we have:

$$(1) \quad G_i(\lambda_{i-1}, s) \geq 0 \quad \text{for } \lambda_i, s \in \mathcal{I},$$

$$(2) \quad \sup_{\lambda \in \mathcal{I}} G_i(\lambda_{i-1}, s) = \sup_{s \in \mathcal{I}} G_i(s, s) \leq \frac{\lambda_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\lambda_i^\rho - \lambda_{i-1}^\rho)]^{\mathcal{K}_i-1}.$$

Proof. The function $G_i(\lambda_{i-1}, s)$ can be written as

$$G_i(\lambda_{i-1}, s) = \frac{\rho^{1-\mathcal{K}_i}}{\Gamma(\mathcal{K}_i)} \begin{cases} \left[\left(\frac{\lambda^\rho - \lambda_{i-1}^\rho}{\lambda_i^\rho - \lambda_{i-1}^\rho} \right)^{\mathcal{K}_i-1} - \left(\frac{\lambda^\rho - s^\rho}{\lambda_i^\rho - s^\rho} \right)^{\mathcal{K}_i-1} \right] (\lambda_{i-1}^\rho - s^\rho)^{\mathcal{K}_i-1} s^{\rho-1}, & \text{for } \lambda_{i-1} \leq s \leq \lambda \leq \lambda_i \\ \left(\frac{\lambda^\rho - \lambda_{i-1}^\rho}{\lambda_i^\rho - \lambda_{i-1}^\rho} \right)^{\mathcal{K}_i-1} (\lambda_i^\rho - s^\rho)^{\mathcal{K}_i-1} s^{\rho-1}, & \text{for } \lambda_{i-1} \leq \lambda \leq s \leq \lambda_i. \end{cases}$$

For convenience, set

$$G_i(\lambda_{i-1}, s) = \frac{\rho^{1-\mathcal{K}_i}}{\Gamma(\mathcal{K}_i)} \begin{cases} G_{1,i}(\lambda_{i-1}, s), & \lambda_{i-1} \leq s \leq \lambda \leq \lambda_i \\ G_{2,i}(\lambda_{i-1}, s), & \lambda_{i-1} \leq \lambda \leq s \leq \lambda_i \end{cases}$$

To show $G_i(\lambda_{i-1}, s) \geq 0$, first note that $G_{2,i}(\lambda_{i-1}, s) \geq 0$, and for $\lambda, s \in (\lambda_{i-1}, \lambda_i]$, we have

$$\begin{aligned} \left(\frac{\lambda^\rho - \lambda_{i-1}^\rho}{\lambda_i^\rho - \lambda_{i-1}^\rho} \right)^{\mathcal{K}_i-1} &= \left[1 - \frac{\lambda_i^\rho - \lambda^\rho}{\lambda_i^\rho - \lambda_{i-1}^\rho} \right]^{\mathcal{K}_i-1}, \\ \left(\frac{\lambda^\rho - s^\rho}{\lambda_i^\rho - s^\rho} \right)^{\mathcal{K}_i-1} &= \left[1 - \frac{\lambda_i^\rho - \lambda^\rho}{\lambda_i^\rho - s^\rho} \right]^{\mathcal{K}_i-1}, \end{aligned}$$

and

$$\lambda_i^\rho - \lambda_{i-1}^\rho \geq \lambda_i^\rho - s^\rho.$$

This implies that $G_{1,i}(\lambda_{i-1}, s) \geq 0$, so part (1) holds.

To show that $G_i(\lambda_{i-1}, s) \leq G_i(s, s)$ for every $s, \lambda_i \in \mathcal{I}$ with $\lambda_i \leq s$, first note that

$$\frac{\partial G_{2,i}(\lambda, s)}{\partial \lambda} = \frac{\rho(\mathcal{K}_i - 1)\lambda^{\rho-1}}{(\lambda_i^\rho - \lambda_{i-1}^\rho)} \left(\frac{\lambda^\rho - \lambda_{i-1}^\rho}{\lambda_i^\rho - \lambda_{i-1}^\rho} \right)^{\mathcal{K}_i-2} (\lambda_i^\rho - s^\rho)^{\mathcal{K}_i-1} s^{\rho-1} \geq 0,$$

so $G_{2,i}$ is increasing on $\lambda_{i-1} \leq \lambda \leq s \leq \lambda_i$. Hence,

$$(3.5) \quad G_{2,i}(\lambda, s) \leq G_i(s, s) \quad \text{for every } s, \lambda \in (\lambda_i, \lambda_i] \quad \text{with } \lambda \leq s.$$

Now for every $s, \lambda \in \mathcal{I}$ with $s \leq \lambda$, we have

$$\begin{aligned} & \frac{\partial G_{1,i}(\lambda, s)}{\partial \lambda} \\ &= \rho(\mathcal{K}_i - 1) \left[\frac{(\lambda^\rho - \lambda_{i-1}^\rho)^{\mathcal{K}_i-2}}{(\lambda^\rho - \lambda_{i-1}^\rho)^{\mathcal{K}_i-1}} - \frac{(\lambda^\rho - s^\rho)^{\mathcal{K}_i-2}}{(\lambda_i^\rho - s^\rho)^{\mathcal{K}_i-1}} \right] (\lambda_i^\rho - s^\rho)^{\mathcal{K}_i-1} (\lambda - s)^{\rho-1} \\ &= \rho(\mathcal{K}_i - 1) \left[\left(\frac{\lambda^\rho - \lambda_{i-1}^\rho}{\lambda^\rho - s^\rho} \right)^{\mathcal{K}_i-2} - \left(\frac{\lambda_i^\rho - \lambda_{i-1}^\rho}{\lambda_i^\rho - s^\rho} \right)^{\mathcal{K}_i-1} \right] \frac{(\lambda^\rho - s^\rho)^{\mathcal{K}_i-2}}{(\lambda_i^\rho - \lambda_{i-1}^\rho)^{\mathcal{K}_i-1}} \\ & \quad \times (\lambda_i^\rho - s^\rho)^{\mathcal{K}_i-1} (\lambda - s)^{\rho-1}. \end{aligned}$$

Clearly,

$$\frac{\lambda^\rho - \lambda_{i-1}^\rho}{\lambda^\rho - s^\rho} \geq 1 \quad \text{and} \quad \frac{\lambda_i^\rho - \lambda_{i-1}^\rho}{\lambda_i^\rho - s^\rho} \geq 1,$$

so

$$\left(\frac{\lambda^\rho - \lambda_{i-1}^\rho}{\lambda^\rho - s^\rho}\right)^{\mathcal{K}_i-2} \leq 1 \quad \text{and} \quad \left(\frac{\lambda_i^\rho - \lambda_{i-1}^\rho}{\lambda_i^\rho - s^\rho}\right)^{\mathcal{K}_i-1} \geq 1.$$

Therefore,

$$\left(\frac{\lambda^\rho - \lambda_{i-1}^\rho}{\lambda^\rho - s^\rho}\right)^{\mathcal{K}_i-2} - \left(\frac{\lambda_i^\rho - \lambda_{i-1}^\rho}{\lambda_i^\rho - s^\rho}\right)^{\mathcal{K}_i-1} \leq 0.$$

This implies

$$\frac{\partial G_{1,i}(\lambda, s)}{\partial \lambda} \leq 0,$$

which means $G_{1,i}$ is decreasing on $\lambda_{i-1} \leq s \leq \lambda \leq \lambda_i$. Therefore, we obtain

$$(3.6) \quad G_{1,i}(\lambda, s) \leq G_{1,i}(s, s) \quad \text{for every } s, \lambda \in (\lambda_{i-1}, \lambda_i] \quad \text{with } s \leq \lambda.$$

In view of (3.5) and (3.6),

$$(3.7) \quad G_i(\lambda, s) \leq G_i(s, s) = \frac{\rho^{1-\mathcal{K}_i}}{\Gamma(\mathcal{K}_i)} \left(\frac{s^\rho - \lambda_{i-1}^\rho}{\lambda_i^\rho - \lambda_{i-1}^\rho}\right)^{\mathcal{K}_i-1} (\lambda_i^\rho - s^\rho)^{\mathcal{K}_i-1} s^{\rho-1}.$$

Finally, we need to show that $G_i(\lambda_{i-1}, s) \leq \frac{\lambda_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\lambda_i^\rho - \lambda_{i-1}^\rho)]^{\mathcal{K}_i-1}$. From (3.7),

$$\begin{aligned} \sup_{s, \lambda \in (\lambda_{i-1}, \lambda_i]} G_i(\lambda, s) &\leq \frac{\rho^{1-\mathcal{K}_i}}{\Gamma(\mathcal{K}_i)} (\lambda_i^\rho - s^\rho)^{\mathcal{K}_i-1} s^{\rho-1} \\ &\leq \frac{\rho^{1-\mathcal{K}_i}}{\Gamma(\mathcal{K}_i)} (\lambda_i^\rho - \lambda_{i-1}^\rho)^{\mathcal{K}_i-1} s^{\rho-1} \\ &\leq \frac{\lambda_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\lambda_i^\rho - \lambda_{i-1}^\rho)]^{\mathcal{K}_i-1}. \end{aligned}$$

This completes the proof of the lemma. □

It is clear that if \hat{w}_i is defined by (3.2), then for each $i = 1, 2, \dots, k$ it is a solution to (1.2).

In order to present our main results, we first list the following conditions that will be needed.

(H1) Let $\mathcal{Q} : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that for all $x_1, x_2, \in \mathbb{R}$, there exists a positive constant $L > 0$ such that

$$|\mathcal{Q}(\lambda, x_1) - \mathcal{Q}(\lambda, x_2)| \leq L|x_1 - x_2|.$$

(H2) $\frac{(\lambda_i - \lambda_{i-1})\lambda_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\lambda_i^\rho - \lambda_{i-1}^\rho)]^{\mathcal{K}_i-1} L < 1$ for each $i = 1, \dots, k$.

Theorem 3.6. *Under conditions (H1) and (H2), the boundary value problem (1.2) has a unique solution in $C([a, b], \mathbb{R})$.*

Proof. : Consider the operator $F : \mathfrak{H}_i \rightarrow \mathfrak{H}_i$ given by

$$F(\hat{\vartheta})(\lambda) = \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda, s) \mathcal{Q}(s, \hat{\vartheta}(s)) ds.$$

Clearly, F is well-defined. For each $i = 1, \dots, n$, let

$$B_{\delta_i} = \{\hat{\vartheta} \in \mathfrak{H}_i : \|\hat{\vartheta}\|_{\mathfrak{H}_i} \leq \delta_i\}.$$

be a non-empty, closed, bounded, convex subset of \mathfrak{H}_i , where

$$(3.8) \quad \delta_i \geq \frac{\frac{(\lambda_i - \lambda_{i-1})\lambda_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\lambda_i^\rho - \lambda_{i-1}^\rho)]^{\mathcal{K}_i-1} \max_{s \in \mathcal{I}} |\mathcal{Q}(s, 0)|}{1 - \frac{(\lambda_i - \lambda_{i-1})\lambda_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\lambda_i^\rho - \lambda_{i-1}^\rho)]^{\mathcal{K}_i-1} L}.$$

We decompose the mapping F into the two maps F_1 and F_2 on B_{δ_i} as follows:

$$\begin{cases} F_1(\hat{\vartheta})(\lambda) = \int_{\lambda_{i-1}}^{\lambda_i} G_{1,i}(\lambda, s) \mathcal{Q}(s, \hat{\vartheta}(s)) ds, \\ F_2(\hat{\vartheta})(\lambda) = \int_{\lambda_{i-1}}^{\lambda_i} G_{2,i}(\lambda, s) \mathcal{Q}(s, \hat{\vartheta}(s)) ds. \end{cases}$$

We will now show that the conditions of Theorem K are satisfied.

Step 1: $F_1(B_{\delta_i}) + F_2(B_{\delta_i}) \subset B_{\delta_i}$. For each $i = 1, \dots, k$, from (H1), we have

$$\begin{aligned} |F_1\hat{\vartheta}(\lambda) + F_2\hat{\vartheta}(\lambda)| &= \left| \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda_{i-1}, s) \mathcal{Q}(s, \hat{\vartheta}(s)) ds \right| \\ &\leq \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda_{i-1}, s) |\mathcal{Q}(s, \hat{\vartheta}(s)) - \mathcal{Q}(s, 0) + \mathcal{Q}(s, 0)| ds, \\ &\leq \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda_{i-1}, s) \left(|\mathcal{Q}(s, \hat{\vartheta}(s)) - \mathcal{Q}(s, 0)| + |\mathcal{Q}(s, 0)| \right) ds, \\ &\leq \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda_{i-1}, s) \left(L \|\hat{\vartheta}\|_{\mathfrak{H}_i} + \max_{s \in \mathcal{I}} |\mathcal{Q}(s, 0)| \right) ds, \\ &\leq \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda_{i-1}, s) \left(L\delta_i + \max_{s \in \mathcal{I}} |\mathcal{Q}(s, 0)| \right) ds, \\ &\leq \frac{(\lambda_i - \lambda_{i-1})\lambda_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\lambda_i^\rho - \lambda_{i-1}^\rho)]^{\mathcal{K}_i-1} \left(L\delta_i + \max_{s \in \mathcal{I}} |\mathcal{Q}(s, 0)| \right) \\ &\leq \delta_i \end{aligned}$$

by (3.8). This shows that condition (a) of Theorem K holds.

Step 2: F_1 is a contraction. For every $i = 1, \dots, n$, we have

$$\begin{aligned}
|(F_1\hat{\vartheta}_2)(\lambda) - (F_1\hat{\vartheta}_1)(\lambda)| &\leq \int_{\lambda_{i-1}}^{\lambda_i} G_{1,i}(\lambda_{i-1}, s) |(\mathcal{Q}(s, \hat{\vartheta}_2(s)) - \mathcal{Q}(s, \hat{\vartheta}_1(s)))| ds, \\
&\leq \int_{\lambda_{i-1}}^{\lambda_i} G_{1,i}(\lambda_{i-1}, s) |(\mathcal{Q}(s, \hat{\vartheta}_2(s)) - \mathcal{Q}(s, \hat{\vartheta}_1(s)))| ds, \\
&\leq L \int_{\lambda_{i-1}}^{\lambda_i} G_{1,i}(\lambda_{i-1}, s) |\hat{\vartheta}_2(s) - \hat{\vartheta}_1(s)| ds, \\
&\leq L \frac{\zeta_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\zeta_i^\rho - \zeta_{i-1}^\rho)]^{\mathcal{K}_i-1} \int_{\lambda_{i-1}}^{\lambda_i} |\hat{\vartheta}_2(s) - \hat{\vartheta}_1(s)| ds, \\
&\leq \frac{(\lambda_i - \lambda_{i-1}) L \zeta_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\zeta_i^\rho - \zeta_{i-1}^\rho)]^{\mathcal{K}_i-1} \|\hat{\vartheta}_2 - \hat{\vartheta}_1\|_{\mathfrak{S}_i} \\
&< \|\hat{\vartheta}_2 - \hat{\vartheta}_1\|_{\mathfrak{S}_i}
\end{aligned}$$

by (H2). Thus, F_1 is a contraction and so condition (b) of Theorem K holds.

Step 3: F_2 is continuous and $F_2(B_{\delta_i})$ is relatively compact. To show that F_2 is continuous, let $\{\hat{\vartheta}_j\}$ be a sequence such that $\hat{\vartheta}_j \rightarrow \hat{\vartheta}$ in B_{δ_i} . Then, for $\lambda \in (\lambda_i, \lambda_{i+1}]$, $i = 1, \dots, k$,

$$\begin{aligned}
|(F_2\hat{\vartheta}_j)(\lambda) - (F_2\hat{\vartheta})(\lambda)| &\leq \int_{\lambda_{i-1}}^{\lambda_i} G_{2,i}(\lambda_{i-1}, s) |(\mathcal{Q}(s, \hat{\vartheta}_j(s)) - \mathcal{Q}(s, \hat{\vartheta}(s)))| ds, \\
&\leq L \int_{\lambda_{i-1}}^{\lambda_i} G_{2,i}(\lambda_{i-1}, s) |\hat{\vartheta}_j(s) - \hat{\vartheta}(s)| ds, \\
&\leq \frac{(\vartheta_i - \vartheta_{i-1}) L \zeta_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\zeta_i^\rho - \zeta_{i-1}^\rho)]^{\mathcal{K}_i-1} \|\hat{\vartheta}_j - \hat{\vartheta}\|_{\mathfrak{S}_i}.
\end{aligned}$$

Applying the Lebesgue Dominated Convergence Theorem yields

$$\|F_2(\hat{\vartheta}_j) - F_2(\hat{\vartheta})\|_{\mathfrak{S}_i} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

To prove that F_2 is equicontinuous, it would suffice to show that it is equicontinuous on each interval $[\lambda_{i-1}, \lambda_i]$, so take $\lambda_1, \lambda_2 \in [\lambda_{i-1}, \lambda_i]$. We have

$$\begin{aligned}
|F_2(\hat{\vartheta})(\lambda_2) - F_2(\hat{\vartheta})(\lambda_1)| &= \left| \int_{\lambda_{i-1}}^{\lambda_i} G_{2,i}(\lambda_2, s) \mathcal{Q}(s, \hat{\vartheta}(s)) ds - \int_{\lambda_{i-1}}^{\lambda_i} G_{2,i}(\lambda_1, s) \mathcal{Q}(s, \hat{\vartheta}(s)) ds \right|, \\
&\leq \int_{\lambda_{i-1}}^{\lambda_i} |G_{2,i}(\lambda_2, s) - G_{2,i}(\lambda_1, s)| |\mathcal{Q}(s, \hat{\vartheta}(s))|_{\mathfrak{S}_i} ds \\
&\leq \left(K \delta_i + \max_{s \in \mathcal{I}} |\mathcal{Q}(s, 0)| \right) \int_{\lambda_{i-1}}^{\lambda_i} |G_{2,i}(\lambda_2, s) - G_{2,i}(\lambda_1, s)| ds.
\end{aligned}$$

As $\lambda_1 \rightarrow \lambda_2$, the right-hand side approaches zero which means that the mapping F_2 is equicontinuous. Now, Step 1 ensures uniform boundedness, so by the Ascoli-Arzelà Theorem, F_2 is relatively compact on B_{δ_i} for each $i = 1, 2, \dots, k$. This shows that (c) of Theorem K holds.

Therefore, by Theorem K, the mapping F has a fixed point and hence the boundary value problem (1.2) has at least one solution in $C([a, b], \mathbb{R})$.

In keeping with the work above, we see that the auxiliary boundary value problem (3.2) has at least one solution in B_{δ_i} for each $i \in \{1, 2, \dots, k\}$. Consequently, the solution to (1.2) can be expressed as

$$\vartheta(\lambda) = \begin{cases} \vartheta_1(\lambda) = \hat{\vartheta}(\lambda), & \lambda \in [a, \lambda_1], \\ \vartheta_2(\lambda) = \begin{cases} 0, & \lambda \in [a, \lambda_1], \\ \hat{\vartheta}_2(\lambda), & \lambda \in (\lambda_1, \lambda_2], \end{cases} \\ \vdots \\ \vartheta_k(\lambda) = \begin{cases} 0, & \lambda \in [a, \lambda_{k-1}], \\ \hat{\vartheta}_k(\lambda), & \lambda \in (\lambda_{k-1}, \lambda_k], \end{cases} \end{cases}$$

To show uniqueness of the solutions, we need to do this on each interval $[\lambda_{i-1}, \lambda_i]$ (see Definition 3.3), so let ϑ_i and ϑ_i^* be solutions of (1.2) on $[\lambda_{i-1}, \lambda_i]$. Then,

$$\begin{aligned} |\vartheta_i(\lambda) - \vartheta_i^*(\lambda)| &= \left| \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda, s) \mathcal{Q}(s, \vartheta(s)) ds - \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda, s) \mathcal{Q}(s, \vartheta^*(s)) ds \right|, \\ &\leq \left| \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda, s) (\mathcal{Q}(s, \vartheta(s)) - \mathcal{Q}(s, \vartheta^*(s))) ds \right|, \\ &\leq \left| \int_{\lambda_{i-1}}^{\lambda_i} G_i(\lambda, s) (\mathcal{Q}(s, \vartheta(s)) - \mathcal{Q}(s, \vartheta^*(s))) ds \right|, \\ &\leq \frac{L \lambda_i^{\rho-1}}{\Gamma(\mathcal{K}_i)} [\rho(\lambda_i^\rho - \lambda_{i-1}^\rho)]^{\mathcal{K}_i-1} \int_{\lambda_{i-1}}^{\lambda_i} |\vartheta(s) - \vartheta^*(s)| ds, \\ &< \int_{\lambda_{i-1}}^{\lambda_i} |\vartheta(s) - \vartheta^*(s)| ds \end{aligned}$$

by (H2), from which the uniqueness of $\vartheta_i(\lambda)$ on $[\lambda_{i-1}, \lambda_i]$ for each $i = 1, 2, \dots, k$ follows. This completes the proof of the theorem. □

4. AN EXAMPLE

As a simple example of our main result obtained in this paper, consider the boundary value problem

$$(4.1) \quad \begin{cases} {}^1\mathfrak{J}^{\mathcal{K}(\lambda)} \vartheta(\lambda) + \mathcal{Q}(\lambda, \vartheta(\lambda)) = 0, & \lambda \in \mathcal{I} = [1, 3], \\ \vartheta(1) = \vartheta(3) = 0, \end{cases}$$

Here we have $\rho = 1$ and

$$\mathcal{K}(\lambda) = \begin{cases} \frac{3}{2}, & \text{if } \lambda \in [1, 2], \\ \frac{5}{2}, & \text{if } \lambda \in (2, 3], \end{cases}$$

so $k = 2$ and our partition is

$$\mathcal{P} = \{[1, 2], (2, 3]\}.$$

We take

$$\mathcal{Q}(\lambda, u) = \frac{1}{2}u$$

and so

$$|\mathcal{Q}(\lambda, w_1(\lambda)) - \mathcal{Q}(\lambda, w_2(\lambda))| = \left| \frac{1}{2}w_1(\lambda) - \frac{1}{2}w_2(\lambda) \right| \leq \frac{1}{2}|w_1(\lambda) - w_2(\lambda)|,$$

which means condition (H1) is satisfied with $L = \frac{1}{2}$.

As to condition (H2), if $i = 1$, then we need $\frac{1}{\Gamma(3/2)}L < 1$, and if $i = 2$, we want $\frac{1}{\Gamma(5/2)}L < 1$, which we have. Therefore, by Theorem K, the problem (4.1) has a unique solution.

In conclusion, as suggestions for future possible research, extending our results here to problems of the form (1.1) as studied by Maazouz and Henderson in [9] would be of interest, as would considering other boundary conditions such as Neumann, Robin, or periodic type. Seeing what results could be obtained by applying other types of fixed point theorems would also be of interest.

REFERENCES

- [1] A. Almeida and S. Samko, Fractional and hypersingular operators in variable exponent spaces on metric measure spaces, *Mediterr. J. Math.* **6** (2009), 215–232.
- [2] C. F. M. Coimbra, Mechanics with variable-order differential operators, *Ann. Phys. (Leipzig)* **12** (2003), 692–703.
- [3] G. Diaz and C. F. M. Coimbra, Nonlinear dynamics and control of a variable order oscillator with application to the van der Pol equation, *Nonlinear Dyn.* **56** (2009), 145–157.
- [4] U. N. Katugampola, New approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.* **6** (2014), 1–15.
- [5] U. N. Katugampola. New approach to a generalized fractional integral, *Appl. Math. Comput.* **218**(2011) 860–865.
- [6] M. A. Krasnosel'skii, Some problems of nonlinear analysis, *Amer. Math. Soc. Transl. Ser 2* **10** (1958), 345–409.
- [7] C. F. Lorenzo and T. T. Hartley, Dynamics and control of initialized fractional-order systems, *Nonlinear Dyn.* **29** (2002), 201–233.
- [8] B. Lupinska, Properties of the Katugampola fractional operators, *Tatra Mt. Math. Publ.* **79** (2021), 135–148.
- [9] K. Maazouz and J. Henderson, Existence results for Katugampola fractional differential equations for boundary value problems, *PanAmer. Math. J.* **30** (2020), 81–98.
- [10] D. S. Oliveira and E. Capelas de Oliveira, Hilfer-Katugampola fractional derivatives, *Comp. Appl. Math.* **37** (2018), 3672–3690.
- [11] H. T. C. Pedro, M. H. Kobayashi, J. M. C. Pereira and C. F. M. Coimbra, Variable order modeling of diffusive-convective effects on the oscillatory flow past a sphere *J. Vib. Control* **14** (2008), 1659–1672.

- [12] S. Rezapour, Z. Bouazza, and M. S. Souid, S. Etemad, and M. K. A. Kaabar, Darbo fixed point criterion on solutions of a Hadamard nonlinear variable order problem and Ulam-Hyers-Rassias stability, *J. Funct. Spaces* **2022** (2022), No. 1769359, 1–12.
- [13] S. G. Samko and B. Ross, Integration and differentiation to a variable fractional-order, *Integral Transforms Spec. Funct.* **1** (1993), 277–300.
- [14] D. R. Smart, *Fixed Point Theorems*, Cambridge Tracts in Math. No. **66**, Cambridge Press, London, 1974.
- [15] S. N. Srivastava, S. Pati, J. R. Graef, A. Domoshnitsky, and S. Padhi, Existence of solution for a Katugampola fractional differential equation using coincidence degree theory, *Mediterr. J. Math.* **21** (2024), No. 123, 1–16.
- [16] H. Sun, A. Chang, Y. Zhang, and W. Chen, *A review on variable-order fractional differential equations: Mathematical foundations, physical models, numerical methods and applications*, *Fract. Calc. Appl. Anal.* **22** (2019), 27–59.
- [17] D. Vivek, K. Kanagarajan and S. Harikrishnan, Theory and analysis of impulsive type pantograph equations with Katugampola fractional derivative, *J. Vibration Test. System Dynam.* **2** (2018), 9–20.
- [18] S. Zhang, *The uniqueness result of solutions to initial value problems of differential equations of variable-order*, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Serie A Mat.* **112** (2018), 407–423.
- [19] S. Zhang and L. Hu, The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order, *AIMS Math.* **5** (2020), 2923–2943.