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NONLOCAL BOUSSINESQ EQUATIONS IN VECTOR VALUED SPACES AND APPLICATIONS

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ABSTRACT. In this paper, the Cauchy problem for linear and nonlinear nonlocal Boussinesq equations is studied. The equation involves convolution terms including abstract operator functions in Fourier type Banach space E. Here, assuming enough smoothness on the initial data and the growth assumptions on operator functions, the local, global existence, uniqueness and regularity properties of solutions are established in terms of fractional powers of given sectorial operator. We can obtain a different classes of nonlocal Boussinesq equations by choosing the space E and linear operator, which occur in a wide variety of physical systems.

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1. Introduction, Definitions and

Background The aim here, is to study the existence, uniqueness and regularity properties of solutions of the Cauchy problem for convolution abstract Boussinesq equation (BE)

$$u_{tt}-\Delta_x u_{tt}+a*\Delta_x u-A*u=[B*f(u)]\,,\ (x,t)\in\mathbb{R}_T^n=\mathbb{R}^n\times(0,T)\,, \tag{1.1}$$

$$u(x,0)=\varphi(x)\,,\ u_t(x,0)=\psi(x)\ \text{ for a.e. } x\in\mathbb{R}^n,$$

where $A=A(x)$ is a linear, $B=B(x)$, $f(u)$ are nonlinear operator functions defined in a Banach space E, a is a complex valued function, $\upsilon*u$ denotes the convolution of $\upsilon=\upsilon(x)$, $u=u(x)$ and $T\in(0,\ \infty]$. Here, Δ_x denotes the Laplace operator with respect to $x\in\mathbb{R}^n$, $\varphi(x)$ and $\psi(x)$ are the given E-valued initial functions.

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First, we consider Cauchy problem for the corresponding linearized problem

$$u_{tt}-\Delta_x u_{tt}+a*\Delta_x u-A*u=g\left(x,t\right),\ \left(x,t\right)\in\mathbb{R}_T^n,$$

$$u\left(x,0\right)=\varphi\left(x\right),\, u_t\left(x,0\right)=\psi\left(x\right)\ \text{for a.e. } x\in\mathbb{R}^n, \tag{1.2}$$

where $g\left(x,t\right)$ is a given E-valued function.

We prove first, the well-posedness of the linear problem (1.2) in E-valued L^p spaces. The L^p-well-posedness of the Cauchy problem (1.1) depends crucially on the presence of a suitable kernel. This fact can be used to obtain the local, global existence, uniqueness and regularity properties of nonlinear problem (1.1).

Note that, the existence, uniqueness of solutions and regularity properties for different type wave equations were considered e.g. in [1-3], [4-7], [8], [10, 11],[14, 15], [17,18, 22] and [28, 33, 34]. Boussinesq type equations occur in a wide variety of physical systems, such as in the propagation of longitudinal deformation waves in an elastic rod, hydro-dynamical process in plasma, in materials science which describe spinodal decomposition and in the absence of mechanical stresses (see [19, 20, 32, 35]).

Cauchy problem for abstract hyperbolic equations were studied e.g. in [3, 4] and [16]. The same time, the nonlocal abstract evolution equations were studied in [24-27]. Unlike previous results, here we determine sufficient conditions on the kernel functions that guarantees global existence and regularity properties of (1.1) in E-valued L^p spaces. We will obtain uniform estimates in terms of fractional powers of operator A. By choosing the space E, operators A and B in (1.1) appropriately, we obtain different classes of nonlocal BEs which occur in application. Let we put $E=l_q$, choose $A=A_1$ and $B=B_1$ as infinite matrices $[a_{ij}]$, $[b_{ij}]$, respectively for $i,j=1,2,...\infty$. Consider the Cauchy problem for infinitely many system of nonlocal linear BEs

$$\Delta_x\partial_t^2 u_i-a*\Delta_x u_i+\sum_{j=1}^{\infty}a_{ij}*u_j=g_i\left(x,t\right), \tag{1.3}$$

$$u_i\left(x,0\right)=\varphi_i\left(x\right),\,\partial_t u_i\left(x,0\right)=\psi_i\left(x\right),$$

and nonlinear nonlocal BEs

$$\Delta_x\partial_t^2 u_i-a*\Delta_x u_i=\sum_{j=1}^{\infty}b_{ij}*f_j\left(u_1,u_2,...\right), \tag{1.4}$$

$$t\in\left(0,T\right),\,x\in\mathbb{R}^n,$$

$$u_i\left(x,0\right)=\varphi_i\left(x\right),\,\partial_t u_i\left(x,0\right)=\psi_i\left(x\right),\,i=1,2,...\infty,$$

where $a_{ij}=a_{ij}\left(x\right)$, $b_{ij}=b_{ij}\left(x\right)$ are complex valued functions, f_i are nonlinear functions and

$$u=\left\{u_i\left(x,t\right)\right\},\varphi=\left\{\varphi_i\left(x\right)\right\},\psi=\left\{\psi_i\left(x\right)\right\}.$$

Then from our results, we obtain the existence, uniqueness and regularity properties of the problems (1.3) and (1.4) in terms of fractional powers of matrix operator A in frame of l_q-valued L^p spaces.

Moreover, let us choose $E = L^{p_1}(0,1)$, $B = B_2(x,y)$ is a bounded function on $\mathbb{R}^n \times (0,1)$ and $A = A_2$ to be degenerated differential operator in $L^{p_1}(0,1)$ defined by

$$D(A_2) = \left\{u \in W_\gamma^{[2],p_1}(0,1),\ \alpha_k u^{[\nu_k]}(0) + \beta_k u^{[\nu_k]}(1) = 0,\ k = 1,2\right\}, \tag{1.5}$$

$$A_2(x)u = d_1(x,y) * u^{[2]} + d_2(x,y) * u^{[1]},\ x \in \mathbb{R}^n,\ y \in (0,1),\ \nu_k \in \{0,1\},$$

where $u^{[i]} = \left(y^\gamma \frac{d}{dy}\right)^i u$ for $0 \leq \gamma < 1 - \frac{1}{p_1}$,$p_1 \in (0,\infty)$, $d_1 = d_1(x,y)$ is a continous, $d_2 = d_2(x,y)$ is a bounded function in $y \in [0,1]$ for a.e. $x \in \mathbb{R}^n$, α_k, β_k are complex numbers and $W_\gamma^{[2],p_1}(0,1)$ is a weighted Sobolev space defined by

$$W_\gamma^{[2],p_1}(0,1) = \{u : u \in L^{p_1}(0,1),\ u^{[2]} \in L^{p_1}(0,1),$$

$$\|u\|_{W_\gamma^{[2],p_1}} = \|u\|_{L^{p_1}} + \left\|u^{[2]}\right\|_{L^{p_1}} < \infty.$$

From our general results we also obtain the existence, uniqueness and regularity properties for the nonlocal mixed problem for nonlocal linear degenerate BE

$$\Delta_x u_{tt} - a * \Delta_x u + d_1(x,y) * u^{[2]} + d_2(x,y) * u^{[1]} = g(x,y,t), \tag{1.6}$$

$$x \in \mathbb{R}^n,\ y \in (0,1),\ t \in (0,T),\ u = u(x,y,t),$$

$$\alpha_{ki} u^{[\nu_k]}(x,0,t) + \beta_{ki} u^{[\nu_k]}(x,1,t) = 0,\ k = 1,2,$$

$$u(x,y,0) = \varphi(x,y),\ u_t(x,y,0) = \psi(x,y),$$

and the same problem for the following nonlinear degenerate BE

$$\Delta_x u_{tt} - a * \Delta_x u + d_1(x,y) * u^{[2]} + d_2(x,y) * u^{[1]} = B_2 * f(u), \tag{1.7}$$

$$x \in \mathbb{R}^n,\ y \in (0,1),\ t \in (0,T),$$

$$\alpha_{ki} u^{[\nu_k]}(x,0,t) + \beta_{ki} u^{[\nu_k]}(x,1,t) = 0,\ k = 1,2,$$

$$u(x,y,0) = \varphi(x,y),\ u_t(x,y,0) = \psi(x,y),$$

where,

$$b\frac{\partial^{[i]}}{\partial y^i} * u = \int\limits_{\mathbb{R}^n} b(x-\xi,y)\frac{\partial^{[i]}}{\partial y^i} u(\xi,y,t)\,d\xi \text{ for } b = b(x,y).$$

Then from our general results we deduced the existence, uniqueness and regularity properties of the problems $(1.6) - (1.7)$ in terms of fractional powers of the operator A_2 defined by (1.5) in frame of $L^{p_1}(0,1)$-valued L^p spaces.

It should be noted that, the regularity properties of nonlinear wave equations in terms of interpolation of spaces are very hard to obtain by the usual methods. For this reason, in the proof we use modern analysis tools like the following:

(1) operator-valued Fourier multiplier theorems in abstract L^p spaces;

(2) Embedding and trace theorems in E-valued Sobolev-Lions and Besov-Lions spaces;

(3) Theory of semigroups of linear operators in Banach spaces;

(4) Perturbation theory of operators;

(5) Interpolation of Banach Spaces, and etc.

A strong mathematical motivation of this work is the following:

(a) the equation is nonlocal, i.e. they contain convolution terms and a variable coefficient operator function (generally, unbounded) in the leading part;

(b) we prove that the linear problem (1.2) is L^p-regular and the corresponding nonlinear problem (1.1) has a unique strong global regular solution;

(c) by choosing the space E and the operators A in (1.1) and (1.2), we obtain wide classes of nonlocal wave equations which occur in application;

The strategy is to express the equation (1.1) as an integral equation. To treat the nonlinearity as a small perturbation of the linear part of the equation, the contraction mapping theorem is used. Also, a priori estimates on L^p norms of solutions of the linearized version are utilized. The key step is the derivation of the uniform estimate for solutions of the linearized convolution wave equation. The methods of harmonic analysis, operator theory, interpolation of Banach spaces and embedding theorems in Sobolev spaces are the main tools implemented to carry out the analysis.

In order to state our results precisely, we introduce some notations and some function spaces.

Let E be a Banach space. $L^p(\Omega; E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_p = \|f\|_{L^p(\Omega;E)} = \left(\int_{\Omega} \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, \; 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\Omega;E)} = ess \sup_{x \in \Omega} \|f(x)\|_E .$$

Let E_1 and E_2 be two Banach spaces. $(E_1, E_2)_{\theta,p}$ for $\theta \in (0,1)$, $p \in [1,\infty]$ denotes the real interpolation spaces defined by K-method [29, §1.3.2]. $L(E_1, E_2)$ will denote the space of all bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ it will be denoted by $L(E)$.

$\mathbb{N}$ denotes the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers. Here,

$$S_{\phi}=\left\{\lambda\in\mathbb{C},\ \left\vert\arg\lambda-\pi\right\vert\leq\pi-\phi,\,0\leq\phi<\pi\right\}.$$

A closed linear operator A is said to be ϕ-sectorial (or sectorial) in a Banach space E with bound $M>0$ if $D\left(A\right)$ and $R\left(A\right)$ are dense on E, $N\left(A\right)=\left\{0\right\}$ and

$$\left\Vert\left(A+\lambda I\right)^{-1}\right\Vert_{L(E)}\leq M\left\vert\lambda\right\vert^{-1}$$

for any $\lambda\in S_{\phi}$, $0\leq\phi<\pi$, where I is an identity operator in E, $L\left(E\right)$ is the space of bounded linear operators in E; $D\left(A\right)$ and $R\left(A\right)$ denote domain and range of the operator A. Sometimes we will put $\left(A+\lambda\right)$ instead of $\left(A+\lambda I\right)$.

It is known that (see e.g.[29, §1.15.1]) there exist the fractional powers A^{θ} of a sectorial operator A. Let $E\left(A^{\theta}\right)$ denote the space $D\left(A^{\theta}\right)$ with the graphical norm

$$\left\Vert u\right\Vert_{E\left(A^{\theta}\right)}=\left(\left\Vert u\right\Vert^{p}+\left\Vert A^{\theta}u\right\Vert^{p}\right)^{\frac{1}{p}},\ 1\leq p<\infty,\ 0<\theta<\infty.$$

Remark 1.1. If A is a ϕ-sectorial operator in a Banach space E, then A generates strongly continuous semigroup $U\left(t\right)=U_{A}\left(t\right)$, $t\geq0$ (see e.g. [11, § 1.5]). Moreover, by reasoning as in [8, Lemma 2.2]), we obtain that

$$\left\Vert U_{A}\left(t\right)\right\Vert_{B(E)}\leq Me^{\omega t},\ \ \left\Vert A^{\gamma}U_{A}\left(t\right)\right\Vert_{L(E)}\leq M\left\vert t\right\vert^{-\gamma},\ t\geq0, \tag{2.1}$$

where $0\leq\gamma<1$ and $\omega\in\mathbb{R}$. Note that if $\phi>\frac{\pi}{2}$, then $U_{A}\left(t\right)$ is a bounded analytic semigroup.

A sectorial operator $A\left(\xi\right)$ for $\xi\in\mathbb{R}^{n}$ is said to be uniformly sectorial in a Banach space E, if $D\left(A\left(\xi\right)\right)$ is independent of ξ and the uniform estimate

$$\left\Vert\left(A+\lambda I\right)^{-1}\right\Vert_{L(E)}\leq M\left\vert\lambda\right\vert^{-1}$$

holds for any $\lambda\in S_{\phi}$.

Let Ω be a domain in $\mathbb{R}^{n}$. $C(\Omega;E)$, $C^{m}(\Omega;E)$ will denote the spaces of E-valued bounded uniformly strongly continuous and m-times continuously differentiable functions on Ω, respectively. $S=S(\mathbb{R}^{n};E)$ denotes E-valued Schwartz class, i.e. the space of all E-valued rapidly decreasing smooth functions on $\mathbb{R}^{n}$ equipped with its usual topology generated by seminorms. $S(\mathbb{R}^{n};\mathbb{C})$ denoted by S. Let $S'(\mathbb{R}^{n};E)$ denote the space of all continuous linear functions from S into E, equipped with the bounded convergence topology. Recall $S(\mathbb{R}^{n};E)$ is norm dense in $L^{p}(\mathbb{R}^{n};E)$ when $1\leq p<\infty$.

Let F, F^{-1} be Fourier and inverse Fourier transforms, defined by

$$\hat{u}\left(\xi\right)=Fu=\left(2\pi\right)^{-\frac{n}{2}}\int\limits_{\mathbb{R}^{n}}e^{-ix\xi}u\left(x\right)dx,\ F^{-1}u=\left(2\pi\right)^{-\frac{n}{2}}\int\limits_{\mathbb{R}^{n}}e^{ix\xi}u\left(\xi\right)d\xi$$

for $u\in S(\mathbb{R}^{n};E)$ and $x,\,\xi\in\mathbb{R}^{n}$.

Let $A(x)$ be a linear operator function $A(x)$ with domain $D(A)$ independent on $x \in \mathbb{R}^n$. The Fourier transformation of $A(x)$ is defined as

$$\langle F A(x) u, \varphi \rangle = \langle A(x) u, F \varphi \rangle \text{ for } u \in D(A)) \text{ and } \varphi \in S(\mathbb{R}^n).$$

(For details see e.g [2, Section 3]), where $\langle f, \varphi \rangle$ denote the value of generalized function f on the $\varphi \in S(\mathbb{R}^n)$.

Let $A = A(x)$ be a linear operator such that $Au \in L^1(\mathbb{R}^n; E)$ for $u \in S(\mathbb{R}^n; D(A))$. The convolution $A * u$ for linear operator A and $u \in S(\mathbb{R}^n; D(A))$ is defined as

$$A * u = \int_{\mathbb{R}^n} A(y) u(x-y) dy \text{ for } u \in S(\mathbb{R}^n; D(A)).$$

By using the Fourier transform in a similar way as in scalar case, we obtain that

$$F(A * u) = \hat{A}(\xi) \hat{u}(\xi), \text{ here } \hat{A}(\xi) = (FA)(\xi).$$

Let m be a positive integer. $W^{m,p}(\Omega; E)$ denotes an E-valued Sobolev space of all functions $u \in L^p(\Omega; E)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L^p(\Omega; E)$ with the norm

$$\|u\|_{W^{m,p}(\Omega;E)} = \|u\|_{L^p(\Omega;E)} + \sum_{k=1}^{n} \left\| \frac{\partial^m u}{\partial x_k^m} \right\|_{L^p(\Omega;E)} < \infty.$$

Let $W^{s,p}(\mathbb{R}^n; E)$ denotes the fractional Sobolev space of order $s \in \mathbb{R}$, that is defined as:

$$W^{s,p}(E) = W^{s,p}(\mathbb{R}^n; E) = \Big\{ u \in S'(\mathbb{R}^n; E),$$
$$\|u\|_{W^{s,p}(E)} = \left\| \mathbb{F}^{-1} \left(I + |\xi|^2 \right)^{\frac{s}{2}} \mathbb{F} u \right\|_{L^p(\mathbb{R}^n;E)} < \infty \Big\}.$$

It is clear that

$$W^{0,p}(\mathbb{R}^n; E) = L^p(\mathbb{R}^n; E), \ p \in [1, \infty].$$

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded into E. Here, $W^{s,p}(\mathbb{R}^n; E_0, E)$ denote the Sobolev-Lions type space i.e.,

$$W^{s,p}(\mathbb{R}^n; E_0, E) = \{ u \in W^{s,p}(\mathbb{R}^n; E) \cap L^p(\mathbb{R}^n; E_0),$$
$$\|u\|_{W^{s,p}(\mathbb{R}^n;E_0,E)} = \|u\|_{L^p(\mathbb{R}^n;E_0)} + \|u\|_{W^{s,p}(\mathbb{R}^n;E)} < \infty \Big\}.$$

In a similar way, we define the following Sobolev-Lions type space:

$$W^{2,s,p}(\mathbb{R}_T^n; E_0, E) = \Big\{ u \in S'(\mathbb{R}_T^n; E_0), \ \partial_t^2 u \in L^p(\mathbb{R}_T^n; E),$$
$$\mathbb{F}_x^{-1} \left(1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{u} \in L^p(\mathbb{R}_T^n; E), \ \|u\|_{W^{2,s,p}\left(\mathbb{R}_T^n;E_0,E\right)} =$$
$$\left\| \partial_t^2 u \right\|_{L^p\left(\mathbb{R}_T^n;E\right)} + \left\| \mathbb{F}_x^{-1} \left(I + |\xi|^2 \right)^{\frac{s}{2}} \hat{u} \right\|_{L^p\left(\mathbb{R}_T^n;E\right)} < \infty \Big\}.$$

Let $1 \leq p \leq q < \infty$. A function $\Psi \in L^{\infty}(\mathbb{R}^n)$ is called a Fourier multiplier from $L^p(\mathbb{R}^n; E)$ to $L^q(\mathbb{R}^n; E)$ if the map $P\colon u \to \mathbb{F}^{-1}\Psi(\xi)\mathbb{F}u$ for $u \in S(\mathbb{R}^n; E)$ is well defined and extends to a bounded linear operator

$$P\colon L^p(\mathbb{R}^n; E) \to L^q(\mathbb{R}^n; E).$$

Let A be a sectorial operator in E. Here,

$$X_p = L^p\left(\mathbb{R}^n; E\right), \ X_p\left(A^{\gamma}\right) = L^p\left(\mathbb{R}^n; E\left(A^{\gamma}\right)\right), \ 1 \leq p, \ q \leq \infty,$$

$$Y^{\ s,p} = Y^{\ s,p}\left(E\right) = W^{s,p}\left(\mathbb{R}^n; E\right), \ Y_q^{s,p}\left(E\right) = Y^{\ s,p}\left(E\right) \cap X_q,$$

$$\left\|u\right\|_{Y_q^{s,p}} = \left\|u\right\|_{W^{s,p}(\mathbb{R}^n;E)} + \left\|u\right\|_{X_q} < \infty,$$

$$W^{s,p}\left(A^{\gamma}\right) = W^{s,p}\left(\mathbb{R}^n; E\left(A^{\gamma}\right)\right), \ 0 < \gamma \leq 1,$$

$$Y^{s,p} = Y^{s,p}\left(A, E\right) = W^{s,p}\left(\mathbb{R}^n; E\left(A\right), E\right), \ Y^{2,s,p} = Y^{2,s,p}\left(A, E\right) =$$

$$W^{2,s,p}\left(\mathbb{R}_T^n; E\left(A\right), E\right), \ Y_q^{s,p}\left(A; E\right) = Y^{s,p}\left(E\right) \cap X_q\left(A\right),$$

$$\left\|u\right\|_{Y_q^{s,p}(A,E)} = \left\|u\right\|_{Y^{s,p}(E)} + \left\|u\right\|_{X_q(A)} < \infty,$$

$$\mathbb{E}_{0p} = \left(Y^{s,p}\left(A, E\right), X_p\right)_{\frac{1}{2p},p}, \ \mathbb{E}_{1p} = \left(Y^{s,p}\left(A, E\right), X_p\right)_{\frac{1+p}{2p},p},$$

$$C^{(i)}\left(Y_1^{s,p}\left(A\right)\right) = C^{(i)}\left(\left[0, T\right]; Y_1^{s,p}\left(A\right)\right), i = 0, 1, 2.$$

Definition 1.1. Let $T > 0$ and $\varphi \in \mathbb{E}_{0p}$, $\psi \in \mathbb{E}_{1p}$. The function $u \in C^2\left(Y_1^{s,p}\left(A\right)\right)$ satisfies of the problem (1.1) is called the continuous solution or the strong solution of (1.1). If $T < \infty$, then $u\left(x, t\right)$ is called the local strong solution of the problem (1.1). If $T = \infty$, then $u\left(x, t\right)$ is called the global strong solution of (1.1).

Definition 1.2. The function $u \in L^{\infty}\left(0, T; X_p\left(A\right) \cap X_1\left(A\right)\right)$ satisfies the equation (1.1) in the sense of $S'(\mathbb{R}^n; E\left(A\right))$ (the space of $E\left(A\right)$-valued tempered distributions) is called the weak solution of (1.1).

Definition 1.3. A Banach space E has Fourier type $r \in [1, 2]$ (or a Fourier type Banach space) provided the Fourier transform $\mathbb{F}$ defines a bounded linear operator from $L^r(\mathbb{R}^n; E)$ to $L^{r'}(\mathbb{R}^n; E)$ for $\frac{1}{r} + \frac{1}{r'} = 1$ (see e.g [13, Remark 2.3]).

Let E_0, E_1be two Banach spaces and $0 < \theta < 1$, $1 \leq p < \infty$. The real interpolation of E_0, E_1 is defined (see [29, § 2.3.2]) as:

$$\begin{aligned} \left(E_0, E\right)_{\theta,p} &= \{ u\colon u = u_0 + u_1, \, u_j \in E_j, \, j = 0, 1, \\ \left\|u\right\|_{\left(E_0,E\right)_{\theta,p}} &= \left\{ \int_0^{\infty} \left[t^{-\theta} K\left(t, u\right)\right]^p \frac{dt}{t} \right\}^{\frac{1}{p}} < \infty \}, \end{aligned}$$

where the functional $K\left(t, u\right)$ defined by

$$K\left(t, u\right) = K\left(t, u, E_0, E\right) = \inf_{u = u_0 + u_1} \left(\left\|u_0\right\|_{E_0} + t\left\|u_1\right\|_{E_1}\right).$$

Remark 1.2. By Fubini's theorem we get $L^p\left(\mathbb{R}_T^n; E\right) = L^p\left(0,T; X_p\right)$, for $X_p = L^p\left(\mathbb{R}^n; E\right)$. Then by definition of spaces $Y^{2,s,p}$, $Y^{s,p} = H^{s,p}\left(\mathbb{R}^n; E\left(A\right), E\right)$, X_p we have

$$Y^{2,s,p} = \Big\{u\colon\ u \in W^{2,p}\left(0,T; Y^{s,p}, X_p\right), \\ \left\|u\right\|_{Y^{2,s,p}} = \left\|u\right\|_{L^p(0,T;Y^{s,p})} + \left\|u^{(2)}\right\|_{L^p(0,T;X_p)}\Big\}.$$

By properties of real interpolation of Banach spaces and interpolation of the intersection of the spaces (see e.g. [29, §1.3]), we obtain

$$\mathbb{E}_{0p} = \left(Y^{s,p}\left(A,E\right) \cap X_p, X_p\right)_{\frac{1}{2p},p} = \left(Y^{s,p}\left(E\right), X_p\right)_{\frac{1}{2p},p} \cap \left(X_p\left(A\right), X_p\right)_{\frac{1}{2p},p} = \\ W^{s\left(1-\frac{1}{2p}\right),p}\left(\mathbb{R}^n; E\right) \cap L^p\left(\mathbb{R}^n; \left(E\left(A\right), E\right)_{\frac{1}{2p},p}\right) = \\ W^{s\left(1-\frac{1}{2p}\right),p}\left(\mathbb{R}^n; \left(E\left(A\right), E\right)_{\frac{1}{2p},p}, E\right).$$

In a similar way, we have

$$\mathbb{E}_{1p} = \left(Y^{s,p}\left(A,E\right) \cap X_p, X_p\right)_{\frac{1+p}{2p},p} = W^{\frac{s(p-1)}{2p},p}\left(\mathbb{R}^n; \left(E\left(A\right), E\right)_{\frac{1+p}{2p},p}, E\right).$$

By J. lions-J. Peetre result (see e.g. [29, §1.8.2]) for $u \in W^{2,p}\left(0,T; Y^{s,p}, X_p\right)$ the trace operator

$$u \to \frac{d^i u}{dt^i}\left(t_0\right) = \frac{\partial^i u}{\partial t^i}\left(., t_0\right)$$

is bounded from $Y^{2,s,p}$ into $C\left(0,T; \left(Y^{s,p}, X_p\right)_{\theta_j,p}\right)$, where

$$\theta_j = \frac{1+jp}{2p},\ j = 0,1.$$

Moreover, if $u\left(x,.\right) \in \left(Y^{s,p}, X_p\right)_{\theta_j,p}$, then under some assumptions that will be stated in the Section 3, $f\left(u\right) \in E$ for all x, $t \in \mathbb{R}_T^n$ and the map $u \to f\left(u\right)$ is bounded from $\left(Y^{s,p}, X_p,\right)_{\frac{1}{2p},p}$ into E. Hence, the nonlinear equation (1.1) is satisfied in the Banach space E. Here, $E\left(A\right)$ denotes a domain of A equipped with graphical norm, $\left(Y^{s,p}, X_p\right)_{\theta,p}$ is a real interpolation space between X_p, $Y^{s,p}$ for $\theta \in (0,1)$, $p \in [1,\infty]$ (see e.g. [29, §1.3]).

Let us introduce a Fourier multiplier theorem in $L\left(E\right)$-valued L^p space (see [13, Theorem 4.3]):

Theorem A. Let X and Y have Fourier type $r \in [1,2]$. If $m \in B_{r,1}^{\frac{n}{r}}\left(\mathbb{R}^n; L\left(X,Y\right)\right)$, then m is a Fourier multiplier from $L^p\left(\mathbb{R}^n; X\right)$ to $L^p\left(\mathbb{R}^n; Y\right)$ for each $p \in [1,\infty]$.

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α. Moreover, for u, $\upsilon > 0$ the relations $u \lesssim \upsilon$, $u \approx \upsilon$ means that there exist positive constants C, C_1, C_2 independent on u and υ such that, respectively

$$u \leq C\upsilon,\ C_1\upsilon \leq u \leq C_2\upsilon.$$

The paper is organized as follows: In Section 1, some definitions and background are given. In Section 2, we obtain the existence of unique solution and a priori estimates for solution of the linearized problem (1.1). In Section 3, we show the existence and uniqueness of local strong solution of the problem (1.1). In Section 4, we show the application of the problem (1.1).

2. **Estimates for linearized equation**

In this section, we make the necessary estimates for solutions of the Cauchy problem for the convolution linear BE (1.2).

Let A be a sectorial operator in Forier type space E and $\hat{X}_p(A^\alpha)$ be $D(A^\alpha)$-value function space with the norm

$$\|u\|_{\hat{X}_p(A^\alpha)} = \|A^\alpha * u\|_{X_p},\ \alpha \in [0,1].$$

Remark 2.1. Let A be a sectorial operator in a Banach space E. In view of interpolation of the domains of sectorial operators (see e.g.[29, §1.8.2]), we have the following relation

$$E\left(A^{1-\theta+\varepsilon}\right) \subset (E(A), E)_{\theta,p} \subset E\left(A^{1-\theta-\varepsilon}\right)$$

for $0 < \theta < 1$ and $0 < \varepsilon < 1-\theta$.

Let A be a generator of a strongly continuous cosine operator function in E (see e.g. [11, §11]) defined by

$$C(t) = \frac{1}{2}\left(e^{itA^{\frac{1}{2}}} + e^{-itA^{\frac{1}{2}}}\right).$$

From the definition of sine operator-function $S(t)$, we have

$$S(t)u = \int_0^t C(\sigma)u d\sigma, \text{ i.e. } S(t)u = \frac{1}{2i}A^{-\frac{1}{2}}\left(e^{itA^{\frac{1}{2}}} - e^{-itA^{\frac{1}{2}}}\right). \tag{2.1}$$

Let $\hat{A}(\xi)$ be the Fourier transformation of $A(x)$, i.e. $\hat{A}(\xi) = \mathbb{F}(A(x))$. We assume that $\hat{A}(\xi)$ is an uniformly sectorial operator in E. Let

$$\eta = \eta(\xi) = \left(1+|\xi|^2\right)^{-\frac{1}{2}}\left[\hat{A}(\xi) + |\xi|^2 \hat{a}(\xi)\right]^{\frac{1}{2}}, \tag{2.2}$$

$$\eta_\pm(\xi) = e^{it\eta(\xi)} \pm e^{-it\eta(\xi)},\ C(t) = C(\xi,t) = \frac{\eta_+(\xi)}{2},$$

$$S(t) = S(\xi,t) = \eta^{-1}(\xi)\frac{\eta_-(\xi)}{2i},\ D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}...\partial x_n^{\beta_n}},\ \beta = (\beta_1, \beta_2, ..., \beta_n).$$

Condition 2.1. Assume: (1) E has Fourier type $r \in (1,\ 2]$; (2) $A = A(x)$ is a linear operator with domain $D(A)$ independent on $x \in \mathbb{R}^n$ such that $Au \in L^1(\mathbb{R}^n; E)$ for $u \in S(\mathbb{R}^n; D(A))$ and $\hat{A}(\xi)$ is an uniformly ϕ-sectorial operator in E for $0 \leq \phi < \pi$

such that $\hat{A}^{\frac{1}{2}}(\xi)$ is a generator of a strongly continuous cosine operator function (see e.g. [11, §11]); (3) $\left\|\hat{A}^{\gamma}u\right\| \lesssim \left\|\hat{A}^{\nu}u\right\|$ for $\gamma \leq \nu$ and $u \in D\left(\hat{A}^{\nu}\right)$; (4) $\hat{A}(\xi)$ is a differentiable operator function with independent on $\xi \in \mathbb{R}^n$ domain $D\left(D^{\beta}\hat{A}(\xi)\right) = D\left(\hat{A}\right) = D(A)$ for $|\beta| > \frac{n}{r}$. Moreover, the following uniform estimate holds

$$\left\|\left[D^{\beta}\hat{A}(\xi)\right]\eta^{-\alpha}(\xi)\right\|_{L(E)} \leq M \text{ for } 0 \leq \alpha < 1 - \frac{1}{2p};$$

(5) $\hat{a} \in C^{(m)}(\mathbb{R}^n)$ such that

$$\hat{a}(\xi) \in S_{\phi},\ \left|D^{\beta}\hat{a}(\xi)\right| \leq C, \tag{2.3}$$

for $m = |\beta| > \frac{n}{r}$ and for all $\xi \in \mathbb{R}^n$;

(6) $\varphi \in \mathbb{E}_{0p}$ and $\psi \in \mathbb{E}_{1p}$ for $p \in [1, \infty]$.

First we need the following lemmas:

Lemma 2.1. Let the assumptions (1) and (2) of Condition 2.1 be satisfied. Then, problem (1.2) has a weak solution.

Proof. By using of the Fourier transform and by Remark 1.1, we get from (1.2):

$$\hat{u}_{tt}(\xi,t) - \eta^2(\xi)\hat{u}(\xi,t) = \left(1+|\xi|^2\right)^{-1}\hat{g}(\xi,t), \tag{2.4}$$

$$\hat{u}(\xi,0) = \hat{\varphi}(\xi),\ \hat{u}_t(\xi,0) = \hat{\psi}(\xi),$$

where $\hat{u}(\xi,t)$ is a Fourier transform of $u(x,t)$ in x, $\eta = \eta(\xi)$ is an operator function in E defined by (2.2), $\hat{\varphi}(\xi)$, $\hat{\psi}(\xi)$ are Fourier transform of φ and ψ, respectively. Since

$$\eta^2(\xi) = \left(1+|\xi|^2\right)^{-1}\left[\hat{A}(\xi) + \hat{a}(\xi)|\xi|^2\right],$$

we have

$$\left[\eta^2(\xi)+\lambda\right]^{-1} = \left\{\left(1+|\xi|^2\right)^{-1}\left[\hat{A}(\xi) + |\xi|^2\hat{a}(\xi)\right] + \lambda\right\}^{-1} \text{ for } \lambda \in S_{\phi_1}.$$

So, in view of the assumptions (1), (2) and by [8, Lemma 2.3], we get the following estimate

$$\left\|\left[\eta^2(\xi)+\lambda\right]^{-1}\right\|_{L(E)} \leq \left(1+|\xi|^2\right)\left\|\left[\hat{A}(\xi) + |\xi|^2\hat{a}(\xi) + \lambda\left(1+|\xi|^2\right)\right]^{-1}\right\|_{L(E)} \leq$$

$$M_1\left(1+|\xi|^2\right)\left||\xi|\left|\hat{a}(\xi)\right| + \lambda\left(1+|\xi|^2\right)\right|^{-1} \leq M_2|\lambda|^{-1}$$

for all $\lambda \in S_{\phi_1}$ with $0 \leq \phi_1 + \phi < \pi$ uniformly with respect λ and $\xi \in \mathbb{R}^n$.

By virtue of [11, §11.2], by (2.2) and in view of the the above uniform estimate, $\eta(\xi)$ is a generator of a strongly continuous uniformly bounded operator functions $C(\xi,t)$, $S(\xi,t)$ in E. Moreover, problem (2.4) has a solution expressed as

$$\hat{u}(\xi,t) = C(\xi,t)\hat{\varphi}(\xi) + S(\xi,t)\hat{\psi}(\xi) + \tag{2.5}$$

$$\int_0^t S(\xi, t-\tau)\left(1+|\xi|^2\right)^{-1}\hat{g}(\xi,\tau)\,d\tau,$$

for all $\xi \in \mathbb{R}^n$, i.e. problem (1.2) has a solution

$$u(x,t) = C_1(t)\varphi + S_1(t)\psi + Q(t)g, \tag{2.6}$$

where $C_1(t)$, $S_1(t)$, $Q(t)$ are linear operator functions defined by

$$C_1(t)\varphi = \mathbb{F}^{-1}\left[C(\xi,t)\hat{\varphi}(\xi)\right],\ S_1(t)\psi = \mathbb{F}^{-1}\left[S(\xi,t)\hat{\psi}(\xi)\right],$$

$$Q(t)g = \mathbb{F}^{-1}\tilde{Q}(\xi,t),\ \tilde{Q}(\xi,t) = \mathbb{F}^{-1}\int_0^t S(\xi,t-\tau)\left(1+|\xi|^2\right)^{-1}\hat{g}(\xi,\tau)\,d\tau.$$

Let

$$Y_0(A^\alpha) = \mathbb{E}_{0p} \cap \hat{X}_1(A^\alpha),\ \|u\|_{Y_0(A^\alpha)} = \|u\|_{\mathbb{E}_{0p}} + \|A^\alpha * u\|_E,$$
$$Y_1(A^\alpha) = \mathbb{E}_{1p} \cap \hat{X}_1(A^\alpha),\ \|u\|_{Y_1(A^\alpha)} = \|u\|_{\mathbb{E}_{1p}} + \|A^\alpha * u\|_E.$$

Theorem 2.1. Assume that the Condition 2.1 is satisfied and $s > \frac{n}{r}\left(\frac{2p}{2p-1}\right)$. Let $0 \leq \alpha < 1 - \frac{1}{2p}$. Then for $\varphi \in Y_0(A^\alpha)$, $\psi \in Y_1(A^\alpha)$ and $g \in L^1(0,T;Y_1^{s,p})$ problem (1.2) has a unique strong solution $u \in C([0,T];X_\infty(A))$. Moreover, the following estimate holds

$$\|A^\alpha * u\|_{X_\infty} + \|A^\alpha * u_t\|_{X_\infty} \leq C_0\left[\|\varphi\|_{Y_0^\alpha(A)} + \|\psi\|_{Y_1^\alpha(A)} + \int_0^t \left(\|g(.,\tau)\|_{Y_1^{s,p}} + \|g(.,\tau)\|_{X_1}\right)d\tau\right], \tag{2.7}$$

uniformly in $t \in [0,T]$, where the constant $C_0 > 0$ depends only on A, the space E and initial data.

Proof. From Lemma 2.1, we get that problem (1.2) has a unique weak solution for $\varphi \in \mathbb{E}_{0p}$, $\psi \in \mathbb{E}_{1p}$ and $g(.,t) \in Y_1^{s,p}$. Let we show that this solution $u = u(x,t)$ is strong and $u \in C([0,T];Y^{s,p}(A;E))$. Let $N \in \mathbb{N}$ and

$$\Pi_N = \{\xi : \xi \in \mathbb{R}^n,\ |\xi| \leq N\},\ \Pi'_N = \{\xi : \xi \in \mathbb{R}^n,\ |\xi| \geq N\}.$$

From (2.6), we deduced that

$$\|A^\alpha * u\|_{X_\infty} \lesssim \left\|\mathbb{F}^{-1}C(\xi,t)\hat{A}^\alpha\hat{\varphi}(\xi)\right\|_{L^\infty(\Pi_N)} + \left\|\mathbb{F}^{-1}S(\xi,t)\hat{A}^\alpha\hat{\psi}(\xi)\right\|_{L^\infty(\Pi_N)} + \left\|\mathbb{F}^{-1}C(\xi,t)\hat{A}^\alpha\hat{\varphi}(\xi)\right\|_{L^\infty(\Pi'_N)} + \left\|\mathbb{F}^{-1}S(\xi,t)\hat{A}^\alpha\hat{\psi}(\xi)\right\|_{L^\infty(\Pi'_N)} + \left\|\mathbb{F}^{-1}\hat{A}^\alpha\tilde{Q}(\xi,t)\right\|_{L^\infty(\Pi_N)} + \tag{2.8}$$

$$\left\|\mathbb{F}^{-1}\hat{A}^{\alpha}\tilde{Q}(\xi,t)\right\|_{L^{\infty}\left(\Pi_N'\right)}.$$

By virtue of Remakes 2.1, 2.2, in view of (2.6) and properties of sectorial operators, we have the following uniform estimate

$$\left\|\mathbb{F}^{-1}\hat{A}^{\alpha}\tilde{Q}(\xi,t)\right\|_{L^{\infty}(\Pi_N)} \leq C\left\|g\right\|_{X_1}.$$

Hence, due to uniform boundedness of operator functions $C(\xi,t)$, $S(\xi,t)$, in view of (2.3) and by Minkowski's inequality for integrals, we get the uniform estimate

$$\left\|\mathbb{F}^{-1}C(\xi,t)\hat{A}^{\alpha}\hat{\varphi}(\xi)\right\|_{L^{\infty}(\Pi_N)} + \left\|\mathbb{F}^{-1}S(\xi,t)\hat{A}^{\alpha}\hat{\psi}(\xi)\right\|_{L^{\infty}(\Pi_N)} \lesssim$$
$$\left[\left\|A^{\alpha}\varphi\right\|_{X_1} + \left\|A^{\alpha}\psi\right\|_{X_1} + \left\|g\right\|_{X_1}\right].$$

Moreover from (2.6), we deduced that

$$\left\|\mathbb{F}^{-1}C(\xi,t)\hat{A}^{\alpha}\hat{\varphi}(\xi)\right\|_{L^{\infty}\left(\Pi_N'\right)} + \left\|\mathbb{F}^{-1}S(\xi,t)\hat{A}^{\alpha}\hat{\psi}(\xi)\right\|_{L^{\infty}} \lesssim$$
$$\left\|\mathbb{F}^{-1}C(\xi,t)\hat{A}^{\alpha}\hat{\varphi}(\xi)\right\|_{L^{\infty}} + \left\|\mathbb{F}^{-1}S(\xi,t)\hat{A}^{\alpha}\hat{\psi}(\xi)\right\|_{L^{\infty}} +$$
$$\left\|\mathbb{F}^{-1}S(\xi,t)\hat{A}^{\alpha}\tilde{Q}(\xi,t)\right\|_{L^{\infty}} \lesssim$$

$$\left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{-\frac{s}{2}}C(\xi,t)\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{A}^{\alpha}\hat{\varphi}(\xi)\right\|_{L^{\infty}} + \tag{2.9}$$
$$\left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{-\frac{s}{2}}S(\xi,t)\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{A}^{\alpha}\hat{\psi}(\xi)\right\|_{L^{\infty}} +$$
$$\left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{-\frac{s}{2}}S(\xi,t)\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{A}^{\alpha}\tilde{Q}(\xi,t)\right\|_{L^{\infty}}.$$

Here, L^{∞} denotes $L^{\infty}(\Omega;E)$. Let us show that $G_i(.,t)$, $V_i(.,t) \in B_{r,1}^{\frac{n}{r}}(\mathbb{R}^n;L(E))$ for $r\in(1,2]$, $i=0,1$ and for all $t\in[0,T]$, where

$$G_0(\xi,t) = \left(1+|\xi|^2\right)^{-\frac{s}{2}}\hat{A}^{\alpha}(\xi)C(\xi,t)\Phi_0(\xi),$$
$$G_1(\xi,t) = \left(1+|\xi|^2\right)^{-\frac{s}{2}}\hat{A}^{\alpha}(\xi)S(\xi,t)\Phi_1(\xi),$$
$$V_0(\xi,t) = \left(1+|\xi|^2\right)^{-\frac{l}{2}}\hat{A}^{\alpha}(\xi)C(\xi,t)\Phi_0(\xi),$$
$$V_1(\xi,t) = \left(1+|\xi|^2\right)^{-\frac{l}{2}}\hat{A}^{\alpha}(\xi)S(\xi,t)\Phi_1(\xi),$$

$$\Phi_0(\xi) = \left[\hat{A}^{1-\frac{1}{2p}-\varepsilon_0} + \left(1+|\xi|^2\right)^{s\left(1-\frac{1}{2p}\right)}\right]^{-1},\ 0<\varepsilon_0<1-\frac{1}{2p}, \tag{2.10}$$

$$\Phi_1(\xi) = \left[\hat{A}^{\frac{1}{2}-\frac{1}{2p}-\varepsilon} + \left(1+|\xi|^2\right)^{s\left(\frac{1}{2}-\frac{1}{2p}\right)}\right]^{-1},\ 0<\varepsilon_1<\frac{1}{2}-\frac{1}{2p},$$

and

$$l = s\left(1-\frac{1}{2p}\right)-\delta, \text{ for a } \delta>0.$$

By embedding properties of E-valued Sobolev and Besov spaces see e.g [2, § 5] the following embedding

$$B_{r,1}^{\frac{n}{r}}\left(\mathbb{R}^{n};L\left(E\right)\right)\subset W^{\sigma,r}\left(\mathbb{R}^{n};L\left(E\right)\right)$$

is contiuous for it $\sigma>\frac{n}{r}$. Hence, it is sufficient to derive that G_i, $V_i\in W_r^{\sigma}\left(\mathbb{R}^{n};L\left(E\right)\right)$ for $\sigma>\frac{n}{r}$. By (2.6) it is clear to see that

$$\frac{\partial}{\partial\xi_k}G_0\left(\xi,t\right)= \tag{2.11}$$

$$-s\xi_k\left(1+|\xi|^2\right)^{-\frac{s}{2}-1}\hat{A}^{\alpha}\left(\xi\right)C\left(\xi,t\right)\Phi_0\left(\xi\right)+$$

$$\left(1+|\xi|^2\right)^{-\frac{s}{2}}\left[\frac{\partial}{\partial\xi_k}\left[\hat{A}^{\alpha}\left(\xi\right)C\left(\xi,t\right)\right]\Phi_0\left(\xi\right)+\hat{A}^{\alpha}\left(\xi\right)C\left(\xi,t\right)\frac{\partial}{\partial\xi_k}\Phi_0\left(\xi\right)\right]=$$

$$-s\xi_k\left(1+|\xi|^2\right)^{-\frac{s}{2}-1}\hat{A}^{\alpha}\left(\xi\right)C\left(\xi,t\right)\Phi_0\left(\xi\right)+\left(1+|\xi|^2\right)^{-\frac{s}{2}}\left\{\left(\hat{A}\left(\xi\right)+\hat{a}\left(\xi\right)|\xi|^2\right)^{-\frac{1}{2}}\times\right.$$

$$\left[\frac{it}{4}\hat{A}^{\alpha}\left(\xi\right)\eta\left(\xi\right)\left(2\xi_k\hat{a}\left(\xi\right)+|\xi|^2\frac{\partial}{\partial\xi_k}\hat{a}\left(\xi\right)+\frac{\partial}{\partial\xi_k}\hat{A}\left(\xi\right)\right)+\right.$$

$$\left.2\xi_k\left(1+|\xi|^2\right)^{-\frac{3}{2}}\left(\hat{A}\left(\xi\right)+\hat{a}\left(\xi\right)|\xi|^2\right)^{\frac{1}{2}}+\alpha C\left(\xi,t\right)\hat{A}^{\alpha-1}\left(\xi\right)\frac{\partial}{\partial\xi_k}\hat{A}\left(\xi\right)\right]\Phi_0\left(\xi\right)-$$

$$\left.2s\xi_k\left(1-\frac{1}{2p}\right)\hat{A}^{\alpha}\left(\xi\right)C\left(\xi,t\right)\left(1+|\xi|^2\right)^{s\left(1-\frac{1}{2p}\right)-1}\Phi_0^{-2}\left(\xi\right)\right\},$$

$$\frac{\partial}{\partial\xi_k}G_1\left(\xi,t\right)=$$

$$-s\xi_k\left(1+|\xi|^2\right)^{-\frac{s}{2}-1}\hat{A}^{\alpha}\left(\xi\right)S\left(\xi,t\right)\Phi_1\left(\xi\right)+$$

$$\left(1+|\xi|^2\right)^{-\frac{s}{2}}\left\{\left[\frac{t}{4}\hat{A}^{\alpha}\left(\xi\right)\eta_+\left(\xi\right)\left(2\xi_k\hat{a}\left(\xi\right)+|\xi|^2\frac{\partial}{\partial\xi_k}\hat{a}\left(\xi\right)+\frac{\partial}{\partial\xi_k}\hat{A}\left(\xi\right)\right)+\right.\right.$$

$$\frac{t}{4i}\hat{A}^{\alpha}\left(\xi\right)\eta_-\left(\xi\right)\left(2\xi_k\hat{a}\left(\xi\right)+|\xi|^2\frac{\partial}{\partial\xi_k}\hat{a}\left(\xi\right)+\frac{\partial}{\partial\xi_k}\hat{A}\left(\xi\right)\right)\eta^{-2}\left(\xi\right)+$$

$$\left.2\xi_k\left(1+|\xi|^2\right)^{-\frac{3}{2}}\left(\hat{A}\left(\xi\right)+\hat{a}\left(\xi\right)|\xi|^2\right)^{\frac{1}{2}}+\alpha S\left(\xi,t\right)\hat{A}^{\alpha-1}\left(\xi\right)\frac{\partial}{\partial\xi_k}\hat{A}\left(\xi\right)\right]\Phi_1\left(\xi\right)-$$

$$\left.2s\xi_k\left(\frac{1}{2}-\frac{1}{2p}\right)\hat{A}^{\alpha}\left(\xi\right)S\left(\xi,t\right)\left(1+|\xi|^2\right)^{s\left(\frac{1}{2}-\frac{1}{2p}\right)-1}\Phi_1^{-2}\left(\xi\right)\right\}.$$

By assumption (3), by (2.3) and (2.10), we have the uniform estmates

$$\left\|\hat{A}^{\alpha}\left(\xi\right)C\left(\xi,t\right)\Phi_0\left(\xi\right)\right\|_{B(E)}\leq C\left\|\hat{A}^{\alpha}\left(\xi\right)\hat{A}^{-\left(1-\frac{1}{2p}-\varepsilon_0\right)}\left(\xi\right)\right\|_{L(E)}\leq C_0,$$

$$\left\|\hat{A}^{\frac{1}{2}}\left(\xi\right)\eta^{-1}\left(\xi\right)\right\|_{L(E)}\left\|\hat{A}^{\alpha}\left(\xi\right)\hat{A}^{-\frac{1}{2}}\left(\xi\right)\Phi_1\left(\xi\right)\right\|_{L(E)}\leq \tag{2.12}$$

$$\left\|\hat{A}^{\alpha}\left(\xi\right)S\left(\xi,t\right)\Phi_1\left(\xi\right)\right\|_{L(E)}\leq C\left\|\hat{A}^{\alpha}\left(\xi\right)\hat{A}^{-\left(1-\frac{1}{2p}-\varepsilon_0\right)}\left(\xi\right)\right\|_{L(E)}\leq C_1.$$

For the proving the inclusion $G_i \in W_2^{\sigma}\left(\mathbb{R}^n; L\left(E\right)\right)$, it is sufficient to show

$$D_{\xi}^{\beta} G_i\left(.,t\right) \in L^r\left(\mathbb{R}^n; L\left(E\right)\right), \ D_{\xi}^{\beta} V_i\left(.,t\right) \in L^r\left(\mathbb{R}^n; L\left(E\right)\right) \tag{2.13}$$

for

$$\beta = \left(\beta_1, \beta_2, ..., \beta_n\right), \ \left|\beta\right| > \frac{n}{r}, \ t \in \left[0, T\right].$$

Indeed, by virtue of (2.1), (2.11)-(2.12) and by assumption on s, we have

$$\int\limits_{\mathbb{R}^n} \left\|G_0\left(\xi,t\right)\right\|_{L(E)}^r d\xi \lesssim \tag{2.14}$$

$$\int\limits_{\mathbb{R}^n} \left(1+\left|\xi\right|^2\right)^{-\frac{rs}{2}} \left\|\hat{A}^{\alpha}\left(\xi\right)\Phi_0\left(\xi\right)C\left(\xi,t\right)\right\|_{L(E)}^r d\xi \lesssim \int\limits_{\mathbb{R}^n} \left(1+\left|\xi\right|^2\right)^{-\frac{rs}{2}} d\xi < \infty,$$

$$\int\limits_{\mathbb{R}^n} \left\|G_1\left(\xi,t\right)\right\|_{L(E)}^r d\xi \lesssim$$

$$\int\limits_{\mathbb{R}^n} \left(1+\left|\xi\right|^2\right)^{-\frac{rs}{2}} \left\|\hat{A}^{\alpha}\left(\xi\right)\Phi_1\left(\xi\right)S\left(\xi,t\right)\right\|_{L(E)}^r d\xi \lesssim \int\limits_{\mathbb{R}^n} \left(1+\left|\xi\right|^2\right)^{-\frac{rs}{2}} d\xi < \infty.$$

By assumption on s and in view of (2.10), $rs\left(1-\frac{1}{2p}\right) > n$. So, by Condition 2.1, we get

$$\int\limits_{\mathbb{R}^n} \left\|V_0\left(\xi,t\right)\right\|_{L(E)}^r d\xi \lesssim \tag{2.15}$$

$$\int\limits_{\mathbb{R}^n} \left(1+\left|\xi\right|^2\right)^{-\frac{rl}{2}} \left\|\hat{A}^{\alpha}\left(\xi\right)\Phi_0\left(\xi\right)C\left(\xi,t\right)\right\|_{L(E)}^r d\xi \lesssim \int\limits_{\mathbb{R}^n} \left(1+\left|\xi\right|^2\right)^{-\frac{rl}{2}} d\xi < \infty,$$

$$\int\limits_{\mathbb{R}^n} \left\|V_1\left(\xi,t\right)\right\|_{L(E)}^r d\xi \lesssim$$

$$\int\limits_{\mathbb{R}^n} \left(1+\left|\xi\right|^2\right)^{-\frac{rl}{2}} \left\|\hat{A}^{\alpha}\left(\xi\right)\Phi_0\left(\xi\right)S\left(\xi,t\right)\right\|_{L(E)}^r d\xi \lesssim \int\limits_{\mathbb{R}^n} \left(1+\left|\xi\right|^2\right)^{-\frac{rl}{2}} d\xi < \infty.$$

First, let we show that

$$\frac{\partial}{\partial \xi_k} G_i\left(.,t\right), \frac{\partial}{\partial \xi_k} V_i\left(.,t\right) \in L^r\left(\mathbb{R}^n; L\left(E\right)\right), \ i = 0, 1, k-1, 2, ...n.$$

By calculating $\frac{\partial}{\partial \xi_k}\Phi_0\left(\xi\right)$, $\frac{\partial}{\partial \xi_k}\Phi_1\left(\xi\right)$ and in view of the assumptions on $\frac{\partial}{\partial \xi_k}\hat{A}\left(\xi\right)$ we have:

$$\hat{A}^{\alpha}\left(\xi\right)\frac{\partial}{\partial \xi_k}\Phi_0\left(\xi\right) \in L\left(E\right), \ \hat{A}^{\alpha}\left(\xi\right)\frac{\partial}{\partial \xi_k}\Phi_1\left(\xi\right) \in L\left(E\right).$$

In view of Condition 2.1, by (2.11)-(2.13) and (2.14) $-$ (2.15), we get

$$\int\limits_{\mathbb{R}^n} \left\|\frac{\partial}{\partial \xi_k} G_0\left(\xi,t\right)\right\|_{L(E)}^r d\xi \lesssim \tag{2.16}$$

$$\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\frac{rs}{2}} \left\|\hat{A}^{\alpha}(\xi)\,\Phi_0 C(\xi,t)\right\|_{L(E)}^r d\xi \lesssim \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\frac{rs}{2}} d\xi < \infty,$$

$$\int_{\mathbb{R}^n} \left\|\frac{\partial}{\partial \xi_k} G_1(\xi,t)\right\|_{L(E)}^r d\xi \lesssim$$

$$\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\frac{rs}{2}} \left\|\hat{A}^{\alpha}(\xi)\,\Phi_1(\xi)\,S(\xi,t)\right\|_{L(E)}^r d\xi \lesssim \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\frac{rs}{2}} d\xi < \infty.$$

Then by differentiating $V_i(\xi,t)$ with respect to ξ_k, we obtain the similar representations (2.11) for $\frac{\partial}{\partial \xi_k} V_i(\xi,t)$ and in a similar way, by using (2.16), we obtain

$$\int_{\mathbb{R}^n} \left\|\frac{\partial}{\partial \xi_k} G_i(\xi,t)\right\|_{L(E)}^r d\xi \leq C,\ i=0.1. \tag{2.17}$$

It is not hard to see that the expressions $D_{\xi}^{\beta} G_i(.,t)$, $D_{\xi}^{\beta} V_i(.,t)$ for $|\beta| > \frac{n}{r}$ include the terms that are multiplication of corresponding terms in (2.11) by bounded operator functions of type

$$D_{\xi}^{\beta}\left[\left(1+|\xi|^2\right)^{-\frac{s}{2}} \hat{A}^{\alpha} \Phi_i^{-k}(\xi)\right].$$

Therefore, in view of estimates (2.16) $-$ (2.17), we obtain (2.13). Hence, by Theorem A the functions $G_i(\xi,t)$, $V_i(.,t)$, $i=0,1$ are $L^p(\mathbb{R}^n;E) \to L^{\infty}(\mathbb{R}^n;E)$ Fourier multipliers. Then by Minkowski's inequality for integrals from (2.3) and (2.8) $-$ (2.10), we have

$$\left\|\mathbb{F}^{-1} C(\xi,t)\,\hat{A}^{\alpha}\hat{\varphi}(\xi)\right\|_{L^{\infty}} + \left\|\mathbb{F}^{-1} S(\xi,t)\,\hat{A}^{\alpha}\hat{\psi}(\xi)\right\|_{L^{\infty}} \lesssim \tag{2.18}$$

$$\left\|\mathbb{F}^{-1} C(\xi,t)\,\eta^{-2}\hat{\varphi}\right\|_{L^{\infty}} + \left\|\mathbb{F}^{-1} S(\xi,t)\,\eta^{-1}\hat{\psi}\right\|_{L^{\infty}} \lesssim$$

$$\left[\|\varphi\|_{\mathbb{E}_{0p}} + \|\psi\|_{\mathbb{E}_{1p}} + \|g\|_{Y^{s,p}}\right].$$

Moreover, by virtue of Remarks 2.1, 2.2 and by reasoning as the above, we have the following estimate

$$\left\|\mathbb{F}^{-1}\hat{A}^{\alpha}\tilde{Q}(\xi,t)\right\|_{X_{\infty}} \leq C \int_0^t \left(\|g(.,\tau)\|_{Y^{s,p}} + \|g(.,\tau)\|_{X_1}\right) d\tau \tag{2.19}$$

uniformly in $t \in [0,T]$. Thus, from (2.6), (2.18) and (2.19), we obtain

$$\|A^{\alpha} * u\|_{X_{\infty}} \leq C\left[\|\varphi\|_{\mathbb{E}_{0p}} + \|A^{\alpha}\varphi\|_{X_1} + \right. \tag{2.20}$$

$$\left. \|\psi\|_{\mathbb{E}_{1p}} + \|A^{\alpha}\psi\|_{X_1} + \int_0^t \left(\|g(.,\tau)\|_{Y^{s,p}} + \|g(.,\tau)\|_{X_1}\right) d\tau\right].$$

By differentiating (2.6) in a similar way, we get

$$\left\|A^{\alpha}*u_{t}\right\|_{X_{\infty}}\leq C\left[\|\varphi\|_{\mathbb{E}_{0p}}+\left\|A^{\alpha}*\varphi\right\|_{X_{1}}+\right. \tag{2.21}$$

$$\left.\left\|A^{\alpha}*\psi\right\|_{\mathbb{E}_{1p}}+\left\|A^{\alpha}*\psi\right\|_{X_{1}}+\int\limits_{0}^{t}\left(\|g\left(.,\tau\right)\|_{Y^{s,p}}+\|g\left(.,\tau\right)\|_{X_{1}}\right)d\tau\right].$$

Then from (2.20) and (2.21) in view of Remarks 2.1, 2.2, we obtain the estimate (2.7).

Let now, we show that problem (1.2) has a unique solution $u\in C\left([0,T];Y^{s,p}\right)$. Let's admit it is the opposite. So let's assume that the problem (1.2) has two solutions

$$u_{1},u_{2}\in C\left([0,T];Y^{s,p}\right).$$

Then by linearity of (1.2), we get that $\upsilon=u_{1}-u_{2}$ is also a solution of the corresponding homogenous equation

$$\left(1-\Delta_{x}\right)u_{tt}-a*\Delta u+A*u=0,$$

$$\upsilon\left(x,0\right)=0,\ \upsilon_{t}\left(x,0\right)=0,\ x\in\mathbb{R}^{n},\ t\in\left(0,T\right).$$

Moreover, by (2.21) we have the following estimate

$$\left\|A^{\alpha}*u\right\|_{X_{\infty}}\leq 0.$$

Since $N\left(A\right)=\{0\}$, the above estmate implies that $\upsilon=0$, i.e. $u_{1}=u_{2}$.

Theorem 2.2. Let the Condition 2.1 holds, $s>\frac{n}{r}\left(\frac{2p}{2p-1}\right)$ and let $0\leq\alpha<1-\frac{1}{2p}$. Then for φ, $\psi\in Y^{s,p}\left(A^{\alpha}\right)$ and $g\in L^{1}\left(0,T;Y^{s,p}\right)$ problem (1.2) has a unique strong solution $u\in C\left([0,T];Y^{s,p}\left(A,E\right)\right)$ and the following uniform estimate holds

$$\left(\left\|A^{\alpha}*u\right\|_{Y^{s,p}}+\left\|A^{\alpha}*u_{t}\right\|_{Y^{s,p}}\right)\leq \tag{2.22}$$

$$C_{0}\left[\|\varphi\|_{Y^{s,p}\left(A^{\alpha}\right)}+\|\psi\|_{Y^{s,p}\left(A^{\alpha}\right)}+\int\limits_{0}^{t}\|g\left(.,\tau\right)\|_{Y^{s,p}}d\tau\right].$$

Proof. From (2.5) and (2.11), we have the following:

$$\left(\left\|\mathbb{F}^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}}\hat{A}^{\alpha}\hat{u}\right\|_{X_{p}}+\left\|\mathbb{F}^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}}\hat{A}^{\alpha}\hat{u}_{t}\right\|_{X_{p}}\right)\leq \tag{2.23}$$

$$C\left\{\left\|\mathbb{F}^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}}C\left(\xi,t\right)\hat{A}^{\alpha}\hat{\varphi}\right\|_{X_{p}}+\left\|\mathbb{F}^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}}\hat{A}^{\alpha}S\left(\xi,t\right)\hat{\psi}\right\|_{X_{p}}+\right.$$

$$\left.\int\limits_{0}^{t}\left\|\left(1+|\xi|^{2}\right)^{\frac{s}{2}}\hat{A}^{\alpha}\tilde{Q}\left(\xi,t\right)\right\|_{X_{p}}d\tau\right\}.$$

By using Theorem A and by reasoning as in Theorem 2.1, we get that $C(\xi,t)$ and $S(\xi,t)$ are Fourier multipliers in $L^p(\mathbb{R}^n;E)$ uniformly with respect to $t\in[0,T]$. So, the estimate (2.23) by using the Minkowski's inequality for integrals implies (2.22).

The uniqueness of (1.2) is obtained by reasoning as in Theorem 2.1.

3. **Local well posedness of IVP for nonlinear nonlocal WE**

In this section, we will show the local existence and uniqueness of solution for the nonlinear problem (1.1).

Let E_0 denote the real interpolation space between $Y^{s,p}(A,E)$ and X_p with $\theta=\frac{1}{2p}$, i.e.

$$\mathbb{E}_{0p}=\left(Y^{s,p}(A,E),X_p\right)_{\frac{1}{2p},p}.$$

Let

$$C^{(i)}\left(Y^{s,p}(A)\right)=C^{(i)}\left([0,T];Y^{s,p}(A,E)\right),\ C^{s,p}(A,E)=C\left([0,T];Y^{s,p}(A,E)\right),\ i=1,2.$$

Condition 3.1. Assume:

(1) the Condition 2.1 holds for $s>\frac{n}{r}\left(\frac{2p}{2p-1}\right)$ and $0\leq\alpha<1-\frac{1}{2p}$;

(2) the kernel $B=B(x)$ is a bounded integrable operator function in E such that

$$B\in L^{\infty}\left(\mathbb{R}^n;L(E)\right)\cap L^{1}\left(\mathbb{R}^n;L(E)\right);$$

(3) the function $u\to f(x,t,u)$: $\mathbb{R}_T^n\times\mathbb{E}_{0p}\to E$ is a measurable in $(x,t)\in\mathbb{R}_T^n$ for $u\in\mathbb{E}_{0p}$. Moreover, $f(x,t,u)$ is continuous in $u\in\mathbb{E}_{0p}$ and $f(x,t,.)\in C^{[s]+1}(\mathbb{E}_{0p};E)$ uniformly with respect to $x\in\mathbb{R}^n$, $t\in[0,T]$, where $\mathbb{R}_T^n=\mathbb{R}^n\times(0,T)$.

Let

$$\hat{Y}_1^{s,p}(A^\alpha;E)=\hat{Y}^{s,p}(A^\alpha;E)\cap X_1(A^\alpha),\ \hat{Y}^{s,p}(A^\alpha;E)=\Big\{u\in Y^{s,p}(A^\alpha;E),$$
$$\|u\|_{\hat{Y}^{s,p}(A^\alpha;E)}=\|A^\alpha*u\|_{X_p}+\left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{u}\right\|_{X_p}<\infty\Big\}.$$

Main aim of this section is to prove the following results:

Theorem 3.1. Let the Condition 3.1 holds. Then there exists a constant $\delta>0$ such that for any $\varphi\in Y_0(A^\alpha)$ and $\psi\in Y_1(A^\alpha)$ satisfying

$$\|\varphi\|_{\mathbb{E}_{0p}}+\|A^\alpha*\varphi\|_{X_1}+\|\psi\|_{\mathbb{E}_{1p}}+\|A^\alpha*\psi\|_{X_1}\leq\delta, \tag{3.1}$$

problem (1.1) has a unique local strang solution $u\in C^1\left(Y_1^{s,p}(A)\right)$. Moreover,

$$\sup_{t\in[0,T]}\left(\|u(.,t)\|_{\hat{Y}_1^{s,p}(A^\alpha,E)}+\|u_t(.,t)\|_{\hat{Y}_1^{s,p}(A^\alpha;E)}\right)\leq C\delta, \tag{3.2}$$

where the constant C depends only on A, E, g, f and initial values.

Proof. By (2.5), ((2.6)) the problem of finding a solution u of (1.1) is equivalent to finding a fixed point of the mapping

$$G(u) = C_1(t)\varphi(x) + S_1(t)\psi(x) + Q(u), \tag{3.3}$$

where $C_1(t)$, $S_1(t)$ are defined by (2.6) and $Q(u)$ is a map defined by

$$Q(u) = -\int_0^t \mathbb{F}^{-1}\left[U(\xi, t-\tau)\left(1+|\xi|^2\right)^{-1}\hat{B}(\xi)\hat{f}(u)(\xi,\tau)\right]d\tau.$$

We define the metric space

$$C(T,A) = C_\delta(T,A) = \left\{u \in C^{s,p}(A,E),\ \|u\|_{C^{s,p}(T,A)} \leq 5C_0\delta\right\}$$

equipped with the norm defined by

$$\|u\|_{C(T,A)} = \sup_{t\in[0,T]}\left[\|A^\alpha * u(.,t)\|_{X_\infty} + \|u(.,t)\|_{Y^{s,p}} + \right.$$

$$\left.\|A^\alpha * u_t(.,t)\|_{X_\infty} + \|u_t(.,t)\|_{Y^{s,p}}\right],$$

where $\delta > 0$ satisfies (3.2) and C_0 is a constant in Theorem 2.1 and 2.2. It is easy to prove that $C(T,A)$ is a complete metric space. From imbedding in Sobolev-Lions space $Y^{s,p}(A,E)$ and by Remark 1.2, we get that $\|u\|_{X_\infty} \leq 1$ if we take that δ is enough small. For $\varphi \in Y_0(A^\alpha)$ and $\psi \in Y_1(A^\alpha)$, let

$$\|\varphi\|_{\mathbb{E}_{0p}} + \|A^\alpha * \varphi\|_{X_1} + \|\psi\|_{\mathbb{E}_{1p}} + \|A^\alpha * \psi\|_{X_1} = \delta.$$

By assumptions, it is easy to see that the map G is well defined for $f \in C^{[s]+1}(\mathbb{E}_{0p}; E)$. By reasoning as in [27, Theorem 4.1], we prove that the G is a contractive map in $C(T,A)$ if δ is a suitable small and G has a unique fixed point in $u \in C(T,A)$, i.e. $u = u(x,t)$ is a solution of (1.1). Let us show that this solution is unique in $C^s(A,E)$. Let u_1, $u_2 \in C^s(A,E)$ are two solution of (1.1). Then for $u = u_1 - u_2$, we have

$$u_{tt} - \Delta_x u_{tt} - a * \Delta u + A * u = B * [f(u_1) - f(u_2)] \tag{3.4}$$

Hence, by Minkowski's inequality for integrals and by Theorem 2.2 from (3.4), we obtain

$$\|u_1 - u_2\|_{Y^{s,p}} \leq C_2(T)\int_0^t \|u_1 - u_2\|_{Y^{s,p}}\,d\tau. \tag{3.5}$$

From (3.4) and Gronwall's inequality, we have $\|u_1 - u_2\|_{Y^{s,p}} = 0$, i.e. problem (1.1) has a unique solution in $C^s(A,E)$.

4. **The Cauchy problem for the infinite system of nonlocal BEs**

Consider the linear problem (1.3). Let

$$l_q = \left\{ u = \{u_j\},\ j = 1, 2, ...N,\ \|u\|_{l_q} = \left(\sum_{j=1}^{N} |u_j|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$N = 1, 2, ...\infty.$$

(see [29, § 1.18]). Let A_1 be the operator in l_p defined by

$$A_1 = [a_{jm}(x)],\ a_{jm} = b_j(x)\, 2^{\sigma m},\ j, m = 1, 2, ...N,\ D(A_1) = l_q^\sigma =$$

$$\left\{ u = \{u_j\},\ \|u\|_{l_q^\sigma} = \left(\sum_{j=1}^{N} 2^{\sigma j} |u_j|^q \right)^{\frac{1}{q}} < \infty \right\},\ \sigma > 0.$$

Let

$$Y^{s,p,\sigma} = W^{s,p}(\mathbb{R}^n; l_q) \cap L^p\left(\mathbb{R}^n; l_q^\sigma\right),\ 1 \leq q \leq \infty,$$

$$W_0(l_q) = W^{s\left(1-\frac{1}{2p}\right),p}(\mathbb{R}^n; l_q) \cap L^p\left(\mathbb{R}^n; l_q^{\sigma\left(1-\frac{1}{2p}\right)}\right).$$

Let $f = \{f_m\}$, $m = 1, 2, ...N$ and

$$\eta_1 = \eta_1(\xi) = \left[\hat{a}(\xi)|\xi|^2 + \hat{A}_1(\xi)\right]^{\frac{1}{2}},$$

$$E_{ip}(l_q) = W^{s(1-\theta_i),p}(\mathbb{R}^n; l_q) \cap L^p\left(\mathbb{R}^n; l_q^{\sigma(1-\theta_i)}\right),$$

where

$$\theta_j = \frac{1+ip}{2p},\ i = 0, 1.$$

From Theorem 2.2 we get the following:

Theorem 4.1. Assume that: (1) $0 \leq \alpha < 1 - \frac{1}{2p}$ and $\varphi, \psi \in Y^{s,p}\left(l_q^{\sigma\alpha}\right)$ for $s > \frac{n}{r}\left(\frac{2p}{2p-1}\right)$; (2) $\hat{b}_j = b_j(\xi)$ are nonnegative bounded differentiable functions on $\mathbb{R}^n$ and $a + \hat{b}_j(\xi) \neq 0$ for $\xi \in \mathbb{R}^n$, $D^\alpha \hat{b}_j$ are uniformly bounded on $\mathbb{R}^n$ for $\beta = (\beta_1, \beta_2, ..., \beta_n)$, $|\beta| > \frac{n}{r}$ and the uniform estimate holds

$$\sum_{j=1}^{\infty} \left|D^\alpha \hat{b}_j(\xi)\right|^2 \left[\hat{a}(\xi)|\xi|^2 + \hat{b}_j(\xi)\right]^{-1} \leq M.$$

Then for $g \in L^1(0, T; Y^{s,p}(l_q))$ problem (1.3) has a unique strong solution $u \in C([0, T]; Y^{s,p}(l_q))$ and the following uniform estimate holds

$$\left(\|A_1^\alpha * u\|_{Y^{s,p}(l_q)} + \|A_1^\alpha * u_t\|_{Y^{s,p}(l_q)}\right) \lesssim \tag{4.1}$$

$$\|\varphi\|_{Y^{s,p}\left(l_q^{\sigma\alpha}\right)} + \|\psi\|_{Y^{s,p}\left(l_q^{\sigma\alpha}\right)} + \int\limits_0^t \|g(.,\tau)\|_{Y^{s,p}(l_q)}\, d\tau.$$

Proof. It is known that l_q is a Fourier type space for $q \in (1,\infty)$ (see e.g [21]). By Remark 2.1, by definition of $W^{s,p}(A_1, l_q)$ and by real interpolation of Banach spaces (see e.g. [29, §1.3, 1.18]), we have

$$D\left(A_1^{\alpha}\right) = l_q^{\sigma\alpha}.$$

By assumptions (1), (2) we obtain that $\hat{A}_1(\xi)$ is uniformly sectorial in l_q, $\eta_1(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$ and

$$\left\| D^{\beta}\left[\hat{A}_1(\xi)\,\eta_1^{-1}(\xi)\right]\right\|_{L(l_q)} \leq M$$

for $\beta = (\beta_1, \beta_2, ..., \beta_n)$, $|\beta| > \frac{n}{r}$. Then by virtue of [3, § 3.14, 3.16] the operator A_1 is generator of bounded cosine function in l_q. Hence, by (4), (5), all conditions of Theorem 2.2 are hold, i,e., we get the conclusion.

Now, consider the nonlinear problem (1.4). From Theorem 3.1 we obtain the following result:

Theorem 4.2. Assume that: (1) $0 \leq \alpha < 1 - \frac{1}{2p}$, $\varphi \in E_{0p}(l_q)$, $\psi \in E_{1p}(l_q)$ and $s > \frac{n}{r}\left(\frac{2p}{2p-1}\right)$; (2) the assumption (2) of the Theorem 4.1 holds;(3) the kernels $b_{mj} = b_{mj}(x)$ are bounded functions such that $b_{mj} \in L^{\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, moreover

$$\sum_{j=1}^{\infty}\left|\hat{b}_{mj}(\xi)\right|^q < \infty \text{ for all } \xi \in \mathbb{R}^n \text{ and for all } m;$$

(4) the function

$$u \to f(x,t,u) : \mathbb{R}^n \times [0,T] \times W_0(l_q) \to l_q$$

is a measurable in $(x,t) \in \mathbb{R}^n \times [0,T]$ for $u \in W_0(l_q)$; Moreover, $f(x,t,u)$ is continuous in $u \in W_0(l_q)$ and $f \in C^{[s]+1}(W_0(l_q); l_q)$ uniformly in $x \in \mathbb{R}^n$, $t \in [0,T]$. Then problem (1.4) has a unique local strong solution

$$u \in C\left([0,\ T_0); Y_{\infty}^{s,p}(A_1, l_q)\right),$$

where T_0 is a maximal time interval that is appropriately small relative to M. Moreover, if

$$\sup_{t\in[0\,,T_0)}\left(\|u\|_{Y_{\infty}^{s,p}\left(\hat{A}_1^{\alpha};l_q\right)} + \|u_t\|_{Y_{\infty}^{s,p}\left(\hat{A}_1^{\alpha};l_q\right)}\right) < \infty,$$

then $T_0 = \infty$.

Proof. It is known that l_q is a Fourier type space (see e.g [21]). By Remark 2.1, by definition of $Y^{s,p}(A_1, l_q)$ and by real interpolation of Banach spaces (see e.g.

[29, §1.3, 1.18]), we have

$$\mathbb{E}_{ip} = \left(W^{s,,p}\left(\mathbb{R}^n; l_q^\sigma, l_q\right), L_p\left(\mathbb{R}^n; l_q\right)_{\theta_i,p}\right) = W^{s(1-\theta_i),p}\left(\mathbb{R}^n; l_q^{\sigma(1-\theta_i)}, l_q\right) =$$

$$W^{s(1-\theta_i),p}\left(\mathbb{R}^n; l_q\right) \cap L^p\left(\mathbb{R}^n; l_q^{\sigma(1-\theta_i)}\right) = E_{0i}\left(l_q\right), \; i = 0,1.$$

By assumptions (1), (2), we obtain that $\hat{A}_1(\xi)$ is uniformly sectorial in l_q, $\eta_1(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$ and

$$\left\| D^\beta \hat{A}_1(\xi)\, \eta_1^{-1}(\xi) \right\|_{L(l_q)} \leq M$$

for $\beta = (\beta_1, \beta_2, ..., \beta_n)$, $|\beta| \leq n$. Then by virtue of [3, § 3.14, 3.16] the operator A_1 is generator of bounded cosine function in l_q. Hence, by (3), (4), all conditions of Theorem 3.1 are hold, i,e., we get the conclusion.

4.2. The mixed problem for degenerate nonlocal BE

Consider the linear problem (1.6). Let

$$Y^{s,p,2} = W^{s,p}\left(\mathbb{R}^n; L^{p_1}(0,1)\right) \cap L^p\left(\mathbb{R}^n; W^{[2],p_1}(0,1)\right), \; 1 \leq p \leq \infty.$$

Let A_2 is the operator in $L^{p_1}(0,1)$ defined by (1.5) and let

$$\eta_2 = \eta_2(\xi) = \left[a\,|\xi|^2 + \hat{A}_2(\xi)\right]^{\frac{1}{2}}.$$

Here,

$$E_{ip}\left(L^{p_1}\right) = W^{[s(1-\theta_i)],p}\left(\mathbb{R}^n; L^{p_1}(0,1)\right) \cap L^p\left(\mathbb{R}^n; W^{[2(1-\theta_i)],p_1}(0,1)\right),$$

where

$$\theta_i = \frac{1+ip}{2p}, \; i = 0,1.$$

Condition 4.2 Assume;

(1) $0 \leq \gamma < 1 - \frac{1}{p_1}$ for $p_1 \in (1,\infty)$ and $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$;

(2) $0 \leq \alpha < 1 - \frac{1}{2p}$, $\varphi \in E_{0p}\left(L^{p_1}\right)$, $\psi \in E_{1p}\left(L^{p_1}\right)$ and $s > \frac{n}{r}\left(\frac{2p}{2p-1}\right)$ for $p \in [1,\infty]$, $p_1 \in (1,\infty)$;

(3) $d_j \in L^\infty\left(\mathbb{R}^n; L^{p_1}(0,1)\right) \cap L^1\left(\mathbb{R}^n; L^{p_1}(0,1)\right)$, $\left\| D^\beta \hat{d}_j(\xi,.) \right\|_{L(L^{p_1})} \leq M$ for $\beta = (\beta_1, \beta_2, ..., \beta_n)$, $|\beta| > \frac{n}{r}$, $j = 1,2$;

(4) $d_1 \in C[0,1]$, $d_1(0,y) = d_1(1,y)$, $d_2(x,y) \in L^\infty(0,1)$ and $|d_2(x,y)| \leq C \left| d_1^{\frac{1}{2}-\mu}(x,y) \right|$ for $0 < \mu < \frac{1}{2}$ and for a.a. $y \in (0,1)$.

Now, we present the following result:

From Theorem 2.1 we obtain the following:

Theorem 4.3. Assume that the Condition 4.2 is satisfied. Then for $\varphi \in Y_0\left(A_2^\alpha\right)$, $\psi \in Y_1\left(A_2^\alpha\right)$ and $g \in L^1\left(0,T; Y_1^{s,p}\left(L^{p_1}\right)\right)$ problem (1.6) has a unique strong solution $u \in C\left([0,T]; X_\infty\left(L^{p_1}\right)\right)$. Moreover, the following estimate holds

$$\left\| A_2^\alpha * u \right\|_{X_\infty} + \left\| A_2^\alpha * u_t \right\|_{X_\infty} \leq C_0 \left[\|\varphi\|_{E_{0p}(L^{p_1})} + \right. \tag{4.2}$$

$$\left.\|\psi\|_{E_{1p}(L^{p_1})}+\int\limits_0^t\left(\|g(.,\tau)\|_{Y_1^{s,p}(L^{p_1})}+\|g(.,\tau)\|_{X_1(L^{p_1})}\right)d\tau\right],$$

uniformly in $t\in[0,T]$, where the constant $C_0>0$ depends only on A_2, the space E and initial data.

Proof. It is known that $L^{p_1}(0,1)$ is a Fourier type space for $p_1\in(1,\infty)$ (see e.g [21]). By Remark 2.1, by definition of $W^{s,p}(A_2,L^{p_1}(0,1))$ and by real interpolation of Banach spaces (see e.g. [29, §1.3, 1.18]), we have

$$\mathbb{E}_{ip}=W^{s,,p}\left(\mathbb{R}^n;W^{[2],,p_1}(0,1),L^{p_1}(0,1),L^p\mathbb{R}^n;L^{p_1}(0,1)\right)_{\theta_i,p}=$$

$$W^{s(1-\theta_i),p}\left(\mathbb{R}^n;W^{[2(1-\theta_i)],p_1}(0,1),L^{p_1}(0,1)\right)=E_{ip}(L^{p_1}),\ i=0,1.$$

By assumptions (1), (2), we obtain that $\hat{A}_2(\xi)$ is uniformly sectorial in $L^{p_1}(0,1)$, $\eta_2(\xi)\neq 0$ for all $\xi\in\mathbb{R}^n$ and

$$\left\|D^\beta\left[\hat{A}_2(\xi)\,\eta_2^{-1}(\xi)\right]\right\|_{L(L^{p_1})}\leq M$$

for $\beta=(\beta_1,\beta_2,...,\beta_n)$, $|\beta|>\frac{n}{r}$. Then by virtue of [3, § 3.14, 3.16] the operator A_2 is generator of bounded cosine function in $L^{p_1}(0,1)$. Hence, by (4), (5) the all conditions of Theorem 2.1 are hold, i,e., we get the conclusion.

Now, consider the nonlinear problem (1.7). From Theorem 3.1 we have the following:

Theorem 4.4. Assume that the Condition 4.1 is satisfied. Moreover,

$$u\to f(x,t,u):\mathbb{R}^n\times[0,T]\times W_0(L^{p_1}(0,1))\to L^{p_1}(0,1)$$

is a measurable in $(x,t)\in\mathbb{R}^n\times[0,T]$ for $u\in W_0(L^{p_1}(0,1))$, $f(x,t,u)$ is continuous in $u\in W_0(L^{p_1}(0,1))$ and

$$f(x,t,.)\in C^{[s]+1}\left(W_0(L^{p_1}(0,1));L^{p_1}(0,1)\right)$$

uniformly with respect to $x\in\mathbb{R}^n$, $t\in[0,T]$. Then problem (1.7) has a unique local strong solution

$$u\in C\left([0,\ T_0);Y_\infty^{s,p}(A_2,L^{p_1}(0,1))\right),$$

where T_0 is a maximal time interval that is appropriately small relative to M. Moreover, if

$$\sup_{t\in[0,T_0)}\left(\|u\|_{Y_\infty^{s,p}\left(\hat{A}_p^\alpha;L^{p_1}(0,1)\right)}+\|u_t\|_{Y_\infty^{s,p}\left(\hat{A}_p^\alpha;L^{p_1}(0,1)\right)}\right)<\infty$$

then $T_0=\infty$.

Proof. Indeed, by reasoning as in Theorem 4.3 and by virtue of [3, § 3.14, 3.16] the operator A_2 is generator of bounded cosine function in $L^{p_1}(0,1)$. Hence, by hypothesis Condition 4.2 and by the second assumption of theorem, we get that all hypothesis of Theorem 3.1 are hold, i.e. we obtain the conclusion.

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