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# MULTIPLE SOLUTIONS FOR A DISCRETE INCLUSION INVOLVING THE $p(k)$-LAPLACE KIRCHHOFF TYPE PROBLEMS

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**ABSTRACT.** In this paper, we prove the existence and multiplicity of solutions for a discrete inclusion involving the $p(k)$-Laplace Kirchhoff type operator in a $T$-dimensional Banach space. We establish the existence of at least one solution and of at least three solutions by using the variational principle for locally Lipschitz functions and the linking theorem under two cases of the nonsmooth potential, nonperiodic and periodic, respectively.

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## 1. INTRODUCTION

In this paper, we consider the following discrete boundary value problem of the $p(k)$-Laplace Kirchhoff type.

$$(1.1)\qquad \begin{cases} -M(\zeta[u])[\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)) - q(k)|u(k)|^{p(k)-2}u(k)] \\ \quad \in \lambda\partial F(k,u(k)), \ \ k\in\mathbb{Z}(1,T), \\ u(0)=u(T+1)=0, \end{cases}$$

where $T\geq 2$ is a fixed positive integer, $\mathbb{Z}(a,b)$ denotes the discrete interval $\{a,a+1,\dots,b-1,b\}$ with $a$ and $b$ integers such that $a<b$, $\triangle u(k)=u(k+1)-u(k)$ is the forward difference operator, $\lambda>0$ is a real parameter, $F:\mathbb{Z}(1,T)\times\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz function with respect to the second variable and $\partial F(k,t)$ is the Clarke subdifferential with respect to the $t$-variable, as defined in [17]. Moreover, $a(k,\cdot), M:[0,\infty)\to[0,\infty)$ are two continuous functions for all $k\in\mathbb{Z}(0,T)$, $t\in[0,\infty)$

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with the function $t \to M(t)$ nondecreasing, $\zeta : \mathbb{R} \to [0, \infty)$ is a functional defined by

$$\zeta[u] = \sum_{k=1}^{T+1} \left( A_0(k-1, |\triangle u(k-1)|) + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right),$$

where $A_0 : \mathbb{Z}(1,T) \times [0,\infty) \to [0,\infty)$ satisfies $A_0(k,t) = \displaystyle\int_0^t a(k,\xi)\xi \, d\xi$, $p : \mathbb{Z}(0,T) \to (1,\infty)$ is a function such that

$$p^- := \min_{k \in \mathbb{Z}(0,T)} p(k) \leq p(k) \leq p^+ := \max_{k \in \mathbb{Z}(0,T)} p(k) \tag{1.2}$$

and the function $q : \mathbb{Z}(1,T) \to (1,\infty)$ is bounded with

$$q^+ := \max_{k \in \mathbb{Z}(1,T)} q(k), \quad \overline{q} := \sum_{k=1}^{T+1} q(k).$$

We assume that $a$ and $M$ satisfy the following conditions.

$(A1)$ $a_1 : \mathbb{Z}(0,T) \to [0,\infty)$ and there exists a constant $a_2 > 0$ such that

$$|a(k,|\xi|)\xi| \leq a_1(k) + a_2|\xi|^{p(k)-1},$$

for all $k \in \mathbb{Z}(0,T)$ and $\xi \in \mathbb{R}$.

$(A2)$ There exists a positive constant $c$ such that

$$\min\left\{ a(k,|\xi|),\ |\xi|\frac{\partial a}{\partial \xi}(k,|\xi|) + a(k,|\xi|) \right\} \geq c|\xi|^{p(k)-2},$$

for all $k \in \mathbb{Z}(0,T)$ and $\xi \in \mathbb{R}$.

$(A3)$ There exist two positive constant $m_0$ and $m_1$ such that

$$m_0 \leq M(t) \leq m_1 \ \text{ for all } \ t \geq 0.$$

$(A4)$ For all $(k,\eta) \in \mathbb{Z}(0,T) \times \mathbb{R}$,

$$0 \leq \Phi(k,\eta)\eta \leq p^+ A_0(k,|\eta|), \ \text{ where } \ \Phi(k,\eta) = a(k,|\eta|)\eta.$$

We also assume that $F(k,t)$ satisfies the following hypotheses.

$(F0)$ There exists a function $r(\cdot) : \mathbb{Z}(1,T) \to (1,\infty)$ such that

$$|w| \leq q(k)|t|^{r(k)-1}, \ \text{ for all } (k,t) \in \mathbb{Z}(1,T) \times \mathbb{R} \text{ and } w \in \partial F(k,t),$$

where $q(\cdot) : \mathbb{Z}(1,T) \to (1,\infty)$ and $1 < r(k) \leq r^+ < p^-$.

$(F1)$ $F(k,0) = 0$ for all $k \in \mathbb{Z}$.

$(F2)$ For all $t \in \mathbb{R}$, $k \mapsto F(k,t)$ is a $T$-periodic function with respect to $k$, i.e. $F(k,t) = F(k+T,t)$, for all $(k,t) \in \mathbb{Z} \times \mathbb{R}$.

$(F3)$ For all $k \in \mathbb{Z}$, $t \mapsto F(k,t)$ is locally Lipschitz.

**Remark 1.1.** As examples of functions $A_0$ and $a$ satisfying the assumptions $(A1)$-$(A4)$, we can give the following.

$(i)$ If we take

$$a(k,|\xi|)=|\xi|^{p(k)-2} \text{ for all } (k,\xi)\in\mathbb{Z}(1,T)\times\mathbb{R},$$

then

$$A_0(k,|\xi|)=\frac{1}{p(k)}|\xi|^{p(k)}.$$

$(ii)$ Now, if we choose

$$a(k,|\xi|)=\left(1+|\xi|^2\right)^{\frac{p(k)-2}{2}} \quad \text{for all } (k,\xi)\in\mathbb{Z}(1,T)\times\mathbb{R},$$

then

$$A_0(k,|\xi|)=\frac{1}{p(k)}\left[\left(1+|\xi|^2\right)^{\frac{p(k)}{2}}-1\right].$$

The nonhomogeneous differential operator

$$\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)),$$

appears on the left-hand side of problem (1.1). Here, $a$ satisfies $(A1)$-$(A4)$. This operator generalizes usual operators with a variable exponent. For example, setting $a(k,|\xi|)=|\xi|^{p(k)-2}$ in problem (1.1) yields the standard $p(k)$-Laplace difference operator

$$\triangle_{p(k-1)}u(k-1):=\triangle\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right).$$

In the case where $a(k,|\xi|)=(1+|\xi|^2)^{\frac{p(k)-2}{2}}$, one has the generalized mean curvature operator

$$\triangle\left(\left(1+|\triangle u(k-1)|^2\right)^{\frac{p(k-1)-2}{2}}\triangle u(k-1)\right).$$

The presence of the nonlocal term $\zeta[u]$ is an important feature of this paper. Problem (1.1) is related to the stationary version of a model called the Kirchhoff equation, introduced in 1876 (see [29]). Kirchhoff's model is given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}=\left(T_0+\frac{Ea}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}, \tag{1.3}$$

where $\rho>0$ is the mass per unit length. $T_0$ is the base tension, $E$ is the Young modulus, $a$ is the area of cross section, and $L$ is the initial length of the string.

Equation (1.3) accounts for the change of tension on the string, caused by its length changing during vibration. Several physicists later considered such equations for research in nonlinear vibrations, both theoretically and experimentally. See, for example, Carrier (see [12] and [13]), Narashima (see [47]), and Oplinger (see [49]). Moreover, Kirchhoff's equation attracted more attention after Lions in 1978 (see [34]) proposed an abstract framework for the problem. This framework deals with the stationary analog of the Kirchhoff-type equation. Many authors have investigated Kirchhoff-type equations, see [3, 14] and the references therein. For recent papers on discrete problems of Kirchhoff type, see [15, 24, 25, 30, 41, 42, 50, 56, 57, 58] and references therein.

In the recent work [43], the present authors have obtained the existence and multiplicity of solutions of the problem (1.1) with $q(k) = 0$, $B_1 t^{\alpha-1} \leq M(t) \leq B_2 t^{\alpha-1}$ (polynomial growth condition) for all $t > 0$ and $\partial F(k, \xi) = \{f(k, \xi)\}$.

The nonlinear difference equations arise in various research fields. The existence and multiplicity of solutions to discrete problems under various boundary value conditions have been widely studied using different methods, such as fixed point theorems, the method of upper and lower solutions, variational methods, critical point theory, minimization methods, linking arguments, and Rabinowitz's global bifurcation theorem. We refer the reader to the monograph by Agarwal [1] and the papers by Cabada, Iannizzotto and Tersian [8], Kone and Ouaro [31], and Mihailescu, Rădulescu and Tersian [38] for more details on difference equations and related applications. We also refer to the recent paper [40] for the applications of variational methods on difference equations. Discrete boundary value problems have been extensively studied in the last few decades. We refer to the recent papers involving the discrete $p$-Laplacian operator and $p(k)$-Laplacian operator [2, 4, 7, 8, 10, 11, 9, 19, 20, 21, 22, 27, 33, 36, 37]. The discrete $p(k)$-Laplacian operator has more complicated nonlinearities than the discrete $p$-Laplacian operator, for example, it is not homogeneous. Problem (1.1) can be seen as a discrete variant of the following variable exponent anisotropic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \left|\frac{\partial u}{\partial x_i}\right|\right) \frac{\partial u}{\partial x_i} + q(x)|u|^{p_i(x)-2}u \in \lambda \partial F(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $\partial F(x, t)$ is the subdifferential with respect to the $t$-variable in the sense of Clarke, $p_i$ and $q$ are continuous functions on $\overline{\Omega}$ such that $q(x) \geq 1$ and $1 < p_i(x)$ for each $x \in \overline{\Omega}$ and every $i \in 1, 2, \ldots, N$, and $\lambda$ is a positive real parameter.

Recently, I.H. Kim and Y.H. Kim [28] analyzed problem (1.4) with $q(x) = 0$, $\partial F(x, t) = \{f(x, u)\}$ with $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. Problems of the form (1.4) without inclusions have been extensively investigated over the past few decades, as they serve as models for a variety of physical phenomena, including those arising in elastic mechanics [59], electrorheological fluids [53, 54], and image restoration [16]. Particularly, discrete inclusions were considered in many papers, see for example [35, 5, 6].

The existence of multiple solutions of discrete inclusions involving the $p(k)$-Laplacian operator was obtained in [20].

In a recent paper [48], by using the nonsmooth three critical points theorem and linking arguments, Ouaro and Zoungrana showed the existence and multiplicity of solutions to discrete inclusions with the $p(k)$-Laplace Kirchhoff type equations.

Inspired by the above-mentioned work, the main goal of this paper is to show the existence and multiplicity of the discrete inclusions of the problem (1.1), via a variational principle for local Lipschitz functions and the linking arguments.

The rest of the paper has the following structure. In Section 2, the variational framework associated with problem (1.1) is established, and some necessary preliminary knowledge on the generalized gradient of the local Lipschitz function is presented. In Section 3, we establish and prove the existence of a nontrivial solution of problem (1.1), by using the Weierstrass theorem. In Section 4, we establish and prove the existence of a nontrivial solution of problem (1.1), by using the nonsmooth mountain pass theorem. In Section 5, we establish and prove the existence of at least two nontrivial solutions of problem (5.1). Finally, in Section 6, we establish and prove the existence of a nontrivial solution corresponding to the non-variational problem (6.1).

## 2. PRELIMINARIES AND SOME LEMMAS

In this section, we present a variational framework for problem (1.1). Specifically, we consider the $T$-dimensional Banach space

$$E=\{u:\mathbb{Z}(0,T+1)\to\mathbb{R}\ \text{ such that }\ u(0)=u(T+1)=0\}$$

endowed with the norm

$$\|u\|=\left(\sum_{k=1}^{T+1}\left(|\triangle u(k-1)|^{p^-}+q(k)|u(k)|^{p^-}\right)\right)^{\frac{1}{p^-}}.$$

On the space $E$, we also introduce the following norm

$$\|u\|_{p^+}=\left(\sum_{k=1}^{T+1}\left(|\triangle u(k-1)|^{p^+}+q(k)|u(k)|^{p^+}\right)\right)^{\frac{1}{p^+}}$$

and the Luxembourg norm

$$\|u\|_{p(\cdot)}=\inf\left\{\nu>0:\ \sum_{k=1}^{T+1}\left(\left|\frac{\triangle u(k-1)}{\nu}\right|^{p(k-1)}+q(k)\left|\frac{u(k)}{\nu}\right|^{p(k)}\right)\leq 1\right\}.$$

Moreover, we use the following inequality.

$$K\|u\|_{p^+}\leq\|u\|\leq 2^{\frac{p^+-p^-}{p^+p^-}}K\|u\|_{p^+}, \tag{2.1}$$

where $K=(\max\{T+1,\overline{q}\})^{\frac{p^+-p^-}{p^+p^-}}$ (see [45]).

Since $E$ is of finite dimension, there exist two constants $0<L_1<L_2$ such that

$$L_1\|u\|_{p(\cdot)}\leq\|u\|\leq L_2\|u\|_{p(\cdot)}. \tag{2.2}$$

Next, let $\varphi : E \to \mathbb{R}$ be given by

$$\varphi(u) = \sum_{k=1}^{T+1} \left[ |\triangle u(k-1)|^{p(k-1)} + q(k)|u(k)|^{p(k)} \right]. \tag{2.3}$$

It is easy to check that for all $u, u_n \in E$ the following relations hold true.

$$\|u\|_{p(\cdot)} < 1 \ \Rightarrow \ \|u\|_{p(\cdot)}^{p^+} \leq \varphi(u) \leq \|u\|_{p(\cdot)}^{p^-}, \tag{2.4}$$

$$\|u\|_{p(\cdot)} > 1 \ \Rightarrow \ \|u\|_{p(\cdot)}^{p^-} \leq \varphi(u) \leq \|u\|_{p(\cdot)}^{p^+}. \tag{2.5}$$

We consider another norm in $E$, that is

$$\|u\|_\infty := \max\{|u(k)| : k \in \mathbb{Z}(1,T)\}, \text{ for all } u \in E.$$

For any $u \in E$, one has

$$\|u\|_\infty \leq \kappa \|u\|, \tag{2.6}$$

where $\kappa := (2T+2)^{1-\frac{1}{p^-}}$ (see [44]).

Now, let $\Phi, \Psi : E \to \mathbb{R}$ be two functionals defined by

$$\Phi(u) = \widehat{M}(\zeta[u]) \tag{2.7}$$

and

$$\Psi(u) = \sum_{k=1}^{T} F(k, u(k)), \tag{2.8}$$

where $\widehat{M}(t) = \displaystyle\int_0^t M(\xi)\, d\xi$.

Thus, for any $u \in E$, the energy functional associated to problem (1.1) is defined as $J : E \to \mathbb{R}$,

$$J(u) := \Phi(u) - \lambda \Psi(u), \ \ \lambda > 0. \tag{2.9}$$

It is easy to see that $\Phi$ is of class $C^1(E, \mathbb{R})$ and its derivative is given by

$$\begin{aligned} \langle \Phi'(u), v \rangle &= M(\zeta[u]) \sum_{k=1}^{T+1} \Big[ a(k-1, |\triangle u(k-1)|) \triangle u(k-1) \triangle v(k-1) \\ &+ q(k)|u(k)|^{p(k)-2} u(k) v(k) \Big], \end{aligned} \tag{2.10}$$

for all $u, v \in E$.

Let $(E, \|\cdot\|)$ be a real Banach space, $E^*$ be its dual and denote $\langle \cdot, \cdot \rangle$ as the duality pairing between $E^*$ and $E$.

A function $J : E \to \mathbb{R}$ is called locally Lipschitz continuous, if for every $u \in E$, we can find a neighbourhood $V_u$ of $u$ and a constant $L_u > 0$ such that

$$|J(v) - J(w)| \leq L_u \|v - w\|, \text{ for all } v, w \in V_u.$$

The generalized directional derivative of $J$ at the point $u \in E$ in the direction $v \in E$ is

$$J^\circ(u; v) = \limsup_{w \to u; \delta \downarrow 0} \frac{J(w + \delta v) - J(w)}{\delta}.$$

The function $v \mapsto J^\circ(u; v)$ is sublinear and continuous. By the Hahn-Banach theorem we know that $J^\circ(u; \cdot)$ is the support function of a nonempty, convex and $w^*$-compact set $\partial J(u) \subseteq E^*$, defined by

$$\partial J(u) = \{u^* \in E^* : \langle u^*, v\rangle \leq J^\circ(u; v), \forall v \in E\}.$$

We say that if $J \in C^1(E, \mathbb{R})$, then $\partial J(u) = \{J'(u)\}$, where $J'(u)$ stands for the first derivative of $J$ at $u$.

A point $u \in E$ is a critical point of the locally Lipschitz continuous function $J$, if $0_{E^*} \in \partial J(u)$, i.e.

$$J^\circ(u; v) \geq 0, \text{ for every } v \in E.$$

It is easily seen that, if $u \in E$ is a local minimum of $J$, then $0_{E^*} \in \partial J(u)$.
A locally Lipschitz function $J : E \to \mathbb{R}$ satisfies the nonsmooth $(PS)$-condition, if any sequence $\{u_n\}_{n \geq 1} \subseteq E$ such that

$$\{J(u_n)\}_{n \geq 1} \text{ is bounded and } \; m(u_n) \to 0 \text{ as } n \to \infty,$$

has a strongly convergent subsequence, where $m(u_n) = \min\{\|u^*\| : \; u^* \in \partial J(u_n)\}$.

We say that a critical point of $J$ is a point $u \in E$ such that

$$\begin{aligned} &-M\left(\zeta[u]\right) \sum_{k=1}^{T+1} \Bigg[a(k-1, |\triangle u(k-1)|)\triangle u(k-1) v(k) \\ &-q(k)|u(k)|^{p(k)-2} u(k) v(k)\Bigg] - \lambda \sum_{k=1}^{T+1} F^\circ(k, u(k)) v(k) \geq 0, \end{aligned} \tag{2.11}$$

which in turn is a solution to (1.1) for any $v \in E$.

**Lemma 2.1.** *(Lebourg's mean value theorem* [39]*). Given the points $x$ and $y$ in $E$ and a real-valued function $J$ which is Lipschitz continuous on an open set containing the segment $(x, y) = \{(1 - t)x + ty : \; t \in (0, 1)\}$, there exist $z = x + t_0(y - x)$, with $0 < t_0 < 1$ and $x^* \in \partial J(z)$ such that*

$$J(y) - J(x) = \langle x^*, y - x\rangle.$$

**Lemma 2.2.** *(Weierstrass theorem* [52]*). Assume that $J$ is a locally Lipschitz functional on a Banach space $E$ and $J : E \to \mathbb{R}$ satisfies.*

*(i) $J$ is weakly lower semicontinuous;*
*(ii) $J$ is coercive.*

*Then, there exists $u^* \in E$ such that $J(u^*) = \min_{u\in E} J(u)$.*

**Lemma 2.3.** *(Nonsmooth mountain pass theorem [32]). Let $E$ be a reflexive Banach space, $J : E \to \mathbb{R}$ a locally Lipschitz functional satisfying the (PS)-condition. Assume that there exist $u_0, u_1 \in E$, $c_0 \in \mathbb{R}$ and $R > 0$ such that $\|u_1 - u_0\| > R$ and*

$$\max\{J(u_0), J(u_1)\} < c_0 = \inf\{J(h) : \|h - u_0\| = R\}.$$

*Then, $J$ has a critical point $u \in E$ with $c = J(u) \geq c_0$, where $c$ is given by*

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s)),$$

*with $\Gamma = \{h \in C([0,1], E) : h(0) = u_0, h(1) = u_1\}$.*

**Lemma 2.4.** *(Linking Theorem [55]). Let $E = V_1 \oplus V_2$ be a real Banach space with $dim V_2 < \infty$. Let $J : E \to \mathbb{R}$ be a local Lipschitz functional satisfying the nonsmooth (PS) condition and such that*

$$J(u) \leq 0 \ (\forall u \in V_2, \|u\| \leq \rho),$$
$$J(u) \geq 0 \ (\forall u \in V_1, \|u\| \leq \rho),$$

*for some $\rho > 0$.*
*Assume that $J$ is bounded from below and $\inf_{u\in E} J(u) < 0$. Then, $J$ has at least two non-zero critical points.*

**Lemma 2.5.** *Suppose that $(F0)$-$(F3)$ and $(A3)$ hold. Then, $J$ is locally Lipschitz in $E$.*

Proof. By virtue of $(A3)$, one has

$$|\Phi(v) - \Phi(w)| = \left|\widehat{M}(\zeta[v]) - \widehat{M}(\zeta[w])\right| = \left|\int_{\zeta[w]}^{\zeta[v]} M(\xi)d\xi\right| \leq m_1(\zeta[v] - \zeta[w]).$$

Note that the quantity $(\zeta[v] - \zeta[w])$ is finite, then we can find $K_1 > 0$ such that

$$(\zeta[v] - \zeta[w]) \leq K_1\|v - w\|.$$

Therefore, for any $v, w \in E$, one has

$$|\Phi(v) - \Phi(w)| \leq m_1 K_1 \|v - w\|. \tag{2.12}$$

Moreover, by $(F0)$ and Lemma 2.1, we have

$$|F(k, v) - F(k, w)| \leq q(k)|\tilde{u}|^{r(k)-1}|v - w|,$$

where $\tilde{u} = tv + (1-t)w$, $t \in (0,1)$.

Thus, by the discrete Hölder inequality (see [23]), we get

$$\begin{aligned} \left| \sum_{k=1}^{T} F(k, v(k)) - \sum_{k=1}^{T} F(k, w(k)) \right| &\leq \sum_{k=1}^{T} q(k) |\tilde{u}(k)|^{r(k)-1} |v(k) - w(k)| \\ &\leq C \| |\tilde{u}|^{r(\cdot)} \|_{r'(\cdot)} \|v - w\|_{r(\cdot)} \leq K_2 \|v - w\|, \end{aligned} \tag{2.13}$$

where $K_2$ is a positive constant.

Then, from (2.12) and (2.13), we obtain

$$\begin{aligned} |J(v) - J(w)| &\leq |\Phi(v) - \Phi(w)| + \lambda |\Psi(v) - \Psi(w)| \\ &\leq m_1 K_1 \|v - w\| + \lambda K_2 \|v - w\| \\ &\leq K \|v - w\|, \end{aligned}$$

with $K = m_1 K_1 + \lambda K_2$. Therefore, $J$ is locally Lipschitz.

**Lemma 2.6.** *Any critical point $u \in E$ of $J$ is a solution of problem* (1.1).

Proof. Suppose that $u \in E$ is a critical point to $J$. Thus, for every $v \in E$, we see that $J^\circ(u; v) \geq 0$. Consequently,

$$\begin{aligned} J(w + \delta v) - J(w) &= \widehat{M}\left(\zeta[w + \delta v]\right) - \widehat{M}\left(\zeta[w]\right) \\ &+ J(w + \delta v) - J(w) - \lambda \sum_{k=1}^{T} \left[ F(k, w(k) + \delta v(k)) - F(k, w(k)) \right]. \end{aligned}$$

Dividing the last equality above by $\delta$ and letting $w \to u$, $\delta \downarrow 0$, one has

$$\begin{aligned} 0 \leq J^\circ(u; v) &= M\left(\eta[u]\right) \sum_{k=1}^{T+1} \Big[ a(k-1, |\triangle u(k-1)|) \triangle u(k-1) \triangle v(k-1) \\ &+ q(k) |u(k)|^{p(k)-2} u(k) v(k) \Big] - \lambda \sum_{k=1}^{T+1} F^\circ(k, u(k)) v(k). \end{aligned}$$

Since $u(0) = u(T+1) = 0$, one has

$$\sum_{k=1}^{T+1} a(k-1, |\triangle u(k-1)|) \triangle u(k-1) \triangle v(k-1) = - \sum_{k=1}^{T+1} \triangle (a(k-1, |\triangle u(k-1)|) \triangle u(k-1)) v(k),$$

then,

$$\begin{aligned} \sum_{k=1}^{T+1} \Big[ &- M(\zeta[u]) [\triangle (a(k-1, |\triangle u(k-1)|) \triangle u(k-1)) \\ &- q(k) |u(k)|^{p(k)-2} u(k)] - \lambda F^\circ(k, u(k)) \Big] v(k) \geq 0. \end{aligned}$$

Therefore, the critical points of $J$ in $E$ are exactly the solutions of problem (1.1).

The following results will also be used in the sequel.

**Lemma 2.7.** *(see* [45]*).*

(*a*) *Let* $u \in E$ *and* $\|u\| > 1$. *Then,*

$$\sum_{k=1}^{T+1} \left[ |\triangle u(k-1)|^{p(k-1)} + q(k)|u(k)|^{p(k)} \right] \geq \|u\|^{p^-} - (1+q^+)(T+1).$$

(*b*) *Let* $u \in E$ *and* $\|u\| < 1$. *Then,*

$$\sum_{k=1}^{T+1} \left[ |\triangle u(k-1)|^{p(k-1)} + q(k)|u(k)|^{p(k)} \right] \geq \frac{2^{\frac{p^- - p^+}{p^-}}}{K^{p^+}} \|u\|^{p^+}.$$

**Lemma 2.8.** *(*[46, Lemma 2]*). Assume that* $(A1)$*-*$(A3)$ *hold. Then, the operator* $\Phi' : E \to E^*$ *is strictly monotone on* $E$ *and is a mapping of type* $(S_+)$*, i.e., if* $u_n \rightharpoonup u$ *in* $E$ *as* $n \to \infty$ *and* $\limsup_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0$*, then* $u_n \to u$ *in* $E$ *as* $n \to \infty$.

**Lemma 2.9.** *(*[46, Lemma 3]*). Under assumptions* $(A1)$*-*$(A3)$*, the functional* $\Phi : E \to \mathbb{R}$ *is weakly lower semicontinuous, i.e.,* $u_n \rightharpoonup u$ *in* $E$ *as* $n \to \infty$ *implies that* $\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n)$.

## 3. EXISTENCE OF SOLUTIONS BY USING WEIERSTRASS THEOREM

In this section, we apply a direct variational approach. We assume that

$(A5)$ There exists a function $r(k) : \mathbb{Z}(1,T) \to (1,\infty)$ with $1 < r^- \leq r^+ < p^- \leq p^+$ such that
$$F(k,t) \geq c_0 |t|^{r(k)}, \text{ for all } (k,t) \in \mathbb{Z}(1,T) \times \mathbb{R},$$
where $c_0$ is a positive constant.

One has the following result.

**Theorem 3.1.** *Assume that* $(A2)$*-*$(A5)$*,* $(F0)$*,* $(F1)$ *and* $(F3)$ *hold with* $r^+ < p^-$. *Then, there exists* $\lambda_0 > 0$ *such that for any* $\lambda > \lambda_0$*, problem* (1.1) *has at least a nontrivial weak solution. When* $p^- = r^+$*, there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$*, problem* (1.1) *has at least one nontrivial weak solution.*

Proof. By Lemma 2.5, $J$ is locally Lipschitz in $E$. From Lemma 2.8, the functional $\Phi$ is weakly lower semicontinuous. Therefore, by the continuity of $\Psi$, we obtain that $J$ is weakly lower semicontinuous. Thus, to prove our result, it suffices to show that $J$ is coercive so that to apply Lemma 2.2. Let $\|u\| > 1$. By condition $(F3)$, one has
$$\frac{d}{dt} F(k,t) \in \partial F(k,t).$$
Moreover, we get by $(F0)$ and $(F1)$ that
$$F(k,t) \quad = \quad F(k,0) + \int_0^t \frac{d}{d\xi} F(k,\xi)\, d\xi$$

$$\leq \frac{q(k)}{r(k)}|t|^{r(k)} \leq q(k)|t|^{r(k)}, \tag{3.1}$$

for all $k \in \mathbb{Z}(1,T)$ and $t \in \mathbb{R}$. Then, it follows from $(A3)$ and (3.1) that

$$\begin{aligned} J(u) &= \widehat{M}(\zeta[u]) - \lambda \sum_{k=1}^{T} F(k, u(k)) \\ &\geq m_0 \int_0^{\zeta[u]} d\xi - \lambda q^+ \sum_{k=1}^{T} |u(k)|^{r(k)}. \end{aligned} \tag{3.2}$$

According to the fact that

$$|u(k)|^{r(k)} \leq |u(k)|^{r^+} + |u(k)|^{r^-}, \text{ for any } k \in \mathbb{Z}(1,T), \tag{3.3}$$

from (2.6) and (3.3), we get

$$\begin{aligned} \sum_{k=1}^{T} |u(k)|^{r(k)} &\leq \sum_{k=1}^{T} |u(k)|^{r^+} + \sum_{k=1}^{T} |u(k)|^{r^-} \\ &\leq \kappa^{r^+} T\|u\|^{r^+} + \kappa^{r^-} T\|u\|^{r^-}. \end{aligned} \tag{3.4}$$

For $u \in E$ with $\|u\| > 1$, by $(A2)$-$(A4)$, (2.2), (2.5), (3.2) and (3.4), we obtain

$$J(u) \geq \frac{m_0 \min\{1, c\}}{p^+ L_2^{p^-}} \|u\|^{p^-} - \lambda q^+ \kappa^{r^+} T \|u\|^{r^+}.$$

Since $p^- > r^+$, then $J(u) \to \infty$ as $\|u\| \to \infty$. Therefore, $J$ is coercive. Hence, by the Weierstrass theorem (Lemma 2.2), we deduce that there exists a global minimizer $u_0 \in E$ such that

$$J(u_0) = \min_{u \in E} J(u),$$

which is a critical point of $J$ and in turn is a weak solution of problem (1.1).

Next, we show that there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, one has $J(u_0) < 0$. Let $d > 1$ be fixed and $k_0 \in \mathbb{Z}(1,T)$. We define a function $w \in E$ by

$$w(k) = \begin{cases} d & \text{if } k = k_0, \\ 0 & \text{if } k \in \mathbb{Z}(1,T) - \{k_0\}. \end{cases}$$

Then, we deduce by conditions $(A3)$ and $(A5)$ that

$$\begin{aligned} J(w) &= m_1 \left( a_1^+ |d| + \frac{a_2}{p(k_0-1)} |d|^{p(k_0-1)} + \frac{q(k_0)}{p(k_0)} |d|^{p(k_0)} \right) - \lambda F(k_0, w(k_0)) \\ &\leq m_1 \left( a_1^+ d + \frac{d^{p^+}}{p^-} (a_2 + q(k_0)) \right) - \lambda c_0 d^{r^-}. \end{aligned}$$

Thus, if we choose

$$\lambda_0 = \frac{m_1 \left[ p^- a_1^+ d^{1-r^-} + d^{p^+ - r^-} (a_2 + q(k_0)) \right]}{p^- c_0},$$

then $J(w) < 0$ for any $\lambda > \lambda_0$. Hence, $J(u_0) < 0$ for any $\lambda > \lambda_0$ and thus $u_0$ is a nontrivial weak solution of problem (1.1) for $\lambda$ large enough.

Now, assume that $p^- = r^+$. Then $J(u) \to \infty$ as $\|u\| \to \infty$ since

$$\frac{m_0 \min\{1, c\}}{p^+ L_2^{p^-}} - \lambda q^+ \kappa^{r^+} T > 0.$$

Putting

$$\lambda^* = \frac{m_0 \min\{1, c\}}{p^+ L_2^{p^-} q^+ \kappa^{r^+} T},$$

then for any $\lambda \in (0, \lambda^*)$, the functional $J$ is coercive.

Using Lemma 2.2, we deduce that there exists $u^* \in E$ a global minimum point of $J$ which is a critical point of $J$ and in turn is a weak solution of problem (1.1).

Next, we will show that $u^*$ is nontrivial. Let $d \in (0, 1)$ be fixed and $k_0 \in \mathbb{Z}(1, T)$. One has

$$J(w) \le m_1 \left( a_1^+ d + \frac{d^{p^-}}{p^-} (a_2 + q(k_0)) \right) - \lambda c_0 d^{r^+}.$$

Then, there exists $\lambda^* > 0$ such that $J(w) < 0$ for any $\lambda \in (0, \lambda^*)$. Therefore, $J(u^*) < 0$ for any $\lambda \in (0, \lambda^*)$ and so $u^*$ is a nontrivial weak solution of problem (1.1).

## 4. EXISTENCE OF SOLUTIONS BY USING NONSMOOTH MOUNTAIN PASS THEOREM

In this section, we investigate the existence of a nontrivial solution by applying Lemma 2.3.

We assume the following additional hypotheses.

$(F4)$ There exist $m_0, m_1, t_1, \theta \in (0, \infty)$ which satisfy $\theta > \frac{m_1}{m_0} p^+$ such that

$$\theta F(k, t) \le F^\circ(k, t)t, \text{ for all } k \in \mathbb{Z}(1, T) \text{ with } |t| > t_1.$$

$(F5)$ $\langle w, t \rangle = \circ(|t|^{p^+})$, as $t \to 0$ uniformly for all $w \in \partial F(k, t)$.

$(F6)$ There exists a function $r(k) : \mathbb{Z}(1, T) \to [2, \infty)$ with $r^- > \frac{m_1}{m_0} p^+$ such that

$$\liminf_{|t| \to \infty} \frac{F(k, t)}{|t|^{r(k)}} \ge 0, \text{ for any } k \in \mathbb{Z}(1, T) \text{ and } t \in \mathbb{R}.$$

One has the following result.

**Theorem 4.1.** *Assume that* $(A1)$-$(A4)$, $(F0)$ *and* $(F4)$-$(F6)$ *hold. Then, the problem* (1.1) *has at least a nontrivial solution for any* $\lambda > 0$.

For the proof of Theorem 4.1, one needs the following lemmas.

**Lemma 4.2.** *Assume that* $(A2)$-$(A4)$ *and* $(F4)$ *hold. Then, for any* $\lambda > 0$, *the functional* $J$ *defined in* (2.9) *satisfies the nonsmooth Palais-Smale condition.*

Proof. Let $\{u_n\}_{n\geq 1}\subseteq E$ be a sequence such that

$$|J(u_n)|\leq M \text{ and } m(u_n)\to 0 \text{ as } n\to\infty, \tag{4.1}$$

for some positive constant $M$. Since $\partial J(u_n)\subseteq E^*$ is weakly compact and the norm functional in a Banach space is weakly lower semicontinuous, from the Weierstrass theorem (Lemma 2.2), we can find $u_n^*\in\partial J(u_n)$ such that $m(u_n)=\|u_n^*\|$, $\forall\, n\geq 1$. Consider the operator $\Phi':E\to E^*$ as in (2.11). Then, it follows from lemmas 2.8 and 2.9 that $\Phi'$ is monotone and semicontinuous, thus, it is maximal monotone (see [26]). Consequently, $u_n^*=\Phi'(u_n)-w_n$ for $n\geq 1$ and $w_n\in\partial F(k,u_n)$. On the other hand, since $\|u_n^*\|\to 0$, we get

$$|\langle u_n^*,u_n\rangle|\leq\varepsilon_n,\varepsilon_n\downarrow 0, \tag{4.2}$$

which means that

$$-m_1p^+\zeta[u_n]+\lambda\sum_{k=1}^{T}w_n(k)u_n(k)\leq\varepsilon_n. \tag{4.3}$$

Since $w_n\geq F^\circ(k,u_n)$, then by (2.9), (4.1) and (4.3), one has

$$\begin{aligned}\theta M+\varepsilon_n &\geq \theta J(u_n)-\langle u_n^*,u_n\rangle\\ &\geq \left(\theta m_0-m_1p^+\right)\zeta[u_n]-\lambda\sum_{k=1}^{T}\left[\theta F(k,u_n(k))-F^\circ(k,u_n(k))u_n(k)\right].\end{aligned}$$

Set

$$M=\max\left\{|F^\circ(k,t)t-\theta F(k,t)|:k\in\mathbb{Z}(1,T),\ |t|\leq t_1\right\}.$$

Then, by $(F4)$, one has

$$\begin{aligned}\left(\theta m_0-m_1p^+\right)\zeta[u_n] &\leq \theta M+\varepsilon_n-\lambda\sum_{k=1,|u_n(k)|>t_1}^{T}\left[F^\circ(k,u_n(k))u_n(k)-\theta F(k,u_n(k))\right]+\lambda MT\\ &\leq \theta M+\varepsilon_n+\lambda MT.\end{aligned}$$

For $n$ large enough, we may assume that $\|u_n\|>1$. From $(A2)$, $(A4)$ and Lemma 2.7$(a)$, we get

$$\left(\theta m_0-m_1p^+\right)\frac{\min\{1,c\}}{p^+}\left(\|u_n\|^{p^-}-(1+q^+)(T+1)\right)\leq\theta M+\varepsilon_n+\lambda MT.$$

Since $\theta>\dfrac{m_1}{m_0}p^+$ and $p^->1$, we infer that $\{u_n\}$ is bounded in $E$. Next, we will show that $u_n\to u$ in $E$ as $n\to\infty$. Since $\{u_n\}$ is bounded in $E$, there exists a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, such that $u_n\rightharpoonup u$ weakly in $E$ as $n\to\infty$. By (4.2), one has

$$\langle\Phi'(u_n),u_n-u\rangle-\lambda\sum_{k=1}^{T}w_n(k)(u_n(k)-u(k))\leq\varepsilon_n,\ \text{ for } n\geq 1.$$

Moreover, $\sum_{k=1}^{T} w_n(k)(u_n(k) - u(k)) \to 0$ as $n \to \infty$, since $\{u_n - u\}_{n\geq 1} \subset E$ and $\{w_n\}_{n\geq 1} \subset E^*$ are bounded. Consequently,

$$\limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle \leq 0.$$

So,

$$\limsup_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0.$$

From Lemma 2.8, one has $u_n \to u$ as $n \to \infty$. Thus, $J$ satisfies the nonsmooth (PS)-condition.

**Lemma 4.3.** *Assume that the hypotheses of Theorem* 4.1 *are satisfied. Then.*

($i$) *There exist* $\varrho, R > 0$ *such that*

$$J(u) \geq \varrho > 0 \ \text{for all } u \in E \text{ with } \|u\| = R.$$

($ii$) *There exists* $e \in E$ *with* $\|e\| > R$ *such that*

$$J(e) < 0.$$

Proof. ($i$) Let us fix $\lambda > 0$ and let

$$0 < \varepsilon < \frac{m_0 \min\{1, c\}}{2^{\frac{p^+}{p^-}} p^+ K^{p^+} \lambda T \kappa^{p^+}}.$$

From Lemma 2.1, we obtain

$$F(k,t) - F(k,0) = \langle w, t\rangle,$$

for some $w \in \partial F(k, \theta t)$ and $\theta \in (0,1)$. Then, condition ($F5$) implies that there exists $\rho \in (0,1)$ such that

$$|F(k,t)| \leq |\langle w, t\rangle| \leq \varepsilon |t|^{p^+}, \tag{4.4}$$

for all $k \in \mathbb{Z}(1,T)$ and all $|t| \leq \rho$.

On the other hand, by condition ($F0$), we deduce that

$$F(k,t) \leq c_\varepsilon q(k)|t|^{r(k)}, \tag{4.5}$$

for all $k \in \mathbb{Z}(1,T)$ and all $|t| > \rho$, where $c_\varepsilon$ is a positive constant.

Combining (4.4) and (4.5), it follows that

$$|F(k,t)| \leq \varepsilon |t|^{p^+} + c_\varepsilon q(k)|t|^{r(k)} \ \text{for all } k \in \mathbb{Z}(1,T) \text{ and } t \in \mathbb{R}.$$

For $u \in E$ such that $\|u\| < 1$, the above inequality, the assumptions ($A2$)-($A4$), Lemma 2.7($b$) and relation (2.6) yield

$$J(u) = \widehat{M}(\zeta[u]) - \lambda \sum_{k=1}^{T} F(k, u(k))$$

$$\begin{aligned} &\geq \frac{2^{\frac{p^- - p^+}{p^-}} m_0 \min\{1,c\}}{p^+ K^{p^+}} \|u\|^{p^+} - \lambda \sum_{k=1}^{T} \left( \varepsilon |u(k)|^{p^+} + c_\varepsilon q(k) |u(k)|^{r(k)} \right) \\ &\geq \frac{2^{\frac{p^- - p^+}{p^-}} m_0 \min\{1,c\}}{p^+ K^{p^+}} \|u\|^{p^+} - \lambda \varepsilon \kappa^{p^+} T \|u\|^{p^+} - \lambda c_\varepsilon q^+ \kappa^{r^-} T \|u\|^{r^-} \\ &\geq \frac{m_0 \min\{1,c\}}{2^{\frac{p^+}{p^-}} p^+ K^{p^+}} \|u\|^{p^+} - c(\lambda,\varepsilon) q^+ \kappa^{r^-} T \|u\|^{r^-}. \end{aligned}$$

Since $p^+ < r^-$, there exist $\varrho, R > 0$ such that $J(u) \geq \varrho > 0$ for all $u \in E$ with $\|u\| = R$. Hence, condition $(i)$ in Lemma 4.2 is fulfilled.

$(ii)$ Fix $\epsilon > 0$. By $(F6)$, there exists $\sigma > 0$ such that

$$F(k,t) \geq \epsilon |t|^{r(k)},$$

for all $(k,t) \in \mathbb{Z}(1,T) \times \mathbb{R}$ with $|t| > \sigma$.
Since $t \to F(k,t) - \epsilon |t|^{r(k)}$ is continuous on $(-\sigma, \sigma)$, there exists $C_\sigma > 0$ such that

$$F(k,t) - \epsilon |t|^{r(k)} \geq -C_\sigma, \text{ for all } k \in \mathbb{Z}(1,T) \text{ and all } t \in (-\sigma,\sigma).$$

Consequently, we get that

$$F(k,t) \geq \epsilon |t|^{r(k)} - C_\sigma, \text{ for all } (k,t) \in \mathbb{Z}(1,T) \times \mathbb{R}. \tag{4.6}$$

Then, for $v \in E \setminus \{0\}$ and $s > 1$, from (4.6), we obtain

$$\begin{aligned} J(sv) &= \widehat{M}(\zeta[sv]) - \lambda \sum_{k=1}^{T} F(k, sv(k)) \\ &\leq m_1 t^{\frac{m_1}{m_0} p^+} (\zeta[v]) - \lambda \epsilon t^{r^-} \sum_{k=1}^{T} |v(k)|^{r(k)} + \lambda C_\sigma T. \end{aligned}$$

Therefore, $J(sv) \to -\infty$ as $t \to \infty$, since $r^- > \dfrac{m_1}{m_0} p^+$. Then, there is some constant $s_0$ such that for $e = s_0 v$ one has $J(e) < 0$. Hence, condition $(ii)$ in Lemma 4.2 is also fulfilled.

Thus, all the conditions of Lemma 2.3 are fulfilled and therefore the problem (1.1) has at least one nontrivial solution.

## 5. EXISTENCE OF SOLUTIONS BY USING THE LINKING THEOREM

In this section, we investigate the existence of multiple $T$-periodic solutions to the following auxiliary problem.

$$\begin{cases} -M(\zeta[u])[\triangle(a(k-1, |\triangle u(k-1)|)\triangle u(k-1)) - q(k)|u(k)|^{p(k)-2} u(k)] \\ \quad \in \lambda \partial F(k, u(k)), \quad k \in \mathbb{Z}, \\ u(k+T) = u(k), \quad k \in \mathbb{Z}. \end{cases} \tag{5.1}$$

Solutions for problem (5.1) will be investigated in the space

$$E_T = \{u = \{u(k)\}_{k\in\mathbb{Z}} : \ u(k) \in \mathbb{R}, \ u(k+T) = u(k), \ k \in \mathbb{Z}\}$$

endowed with the Euclidean norm

$$\|u\|_e := \left(\sum_{k=1}^{T} |u(k)|^2\right)^{\frac{1}{2}}$$

with which $E_T$ becomes an Hilbert space. For each $a \in \mathbb{R}$, let

$$W_a := \{u = \{u(k)\}_{k\in\mathbb{Z}} : \ u(k) = a \in \mathbb{R}, \ k \in \mathbb{Z}\} \text{ and } Y_a := W_a^{\perp}.$$

Assuming that for any $a \in \mathbb{R}$,

$$E_T = Y_a \oplus W_a.$$

However, on the space $Y_a$ we will also consider the following norm

$$\|u\|_{p^+} = \left(\sum_{k=1}^{T} \left(|\triangle u(k-1)|^{p^+} + q(k)|u(k)|^{p^+}\right)\right)^{\frac{1}{p^+}}.$$

The latest norm is obviously not a norm on $E_T$.
For any given $\lambda > 0$, we define the energy functional $J_T : E_T \to \mathbb{R}$ corresponding to (5.1) by

$$J_T = \Phi - \lambda\Psi, \tag{5.2}$$

with $\Phi$ given by (2.7) and $\Psi$ in (2.8).
Arguing as in the proof of Lemma 2.5, it is easy to verify that $J_T$ is the nonsmooth Lipschitz energy functional corresponding to problem (5.1).

Let us assume that.

$(F7)$ There exist $T$-periodic functions $r : \mathbb{Z} \to [2, \infty)$, $\beta : \mathbb{Z} \to (0, \infty)$ and a number $R \geq 1$ (sufficiently large) such that

$$w \geq \beta(k)|t|^{r(k)-1}, \text{ for all } (k,t) \in \mathbb{Z}\times\mathbb{R} \text{ and } w \in \partial F(k,t) \text{ with } |t| \geq R.$$

$(F8)$ $\limsup\limits_{|t|\to 0} \dfrac{\langle w, t\rangle}{|t|^{r^-}} \leq 0$ uniformly for all $w \in \partial F(k,t)$, where $r^- := \min\limits_{k\in\mathbb{Z}(1,T)} r(k)$.

$(F9)$ There exist $f > e > d > 0$ such that
 (i) $\langle w, t\rangle \geq 0$ for all $w \in \partial F(k,t)$ with $|t| \leq d$;
 (ii) $\langle w, t\rangle < 0$ for all $w \in \partial F(k,t)$ with $e \leq |t| \leq f$.

Now, we recall some auxiliary results that we use throughout the paper.

**Lemma 5.1.** *(see [18]). The following properties hold.*

$(i)$ *For every $s > 0$,*

$$\sum_{k=1}^{T} |u(k)|^s \leq T\|u\|_e^s \text{ for all } u \in E_T.$$

(*ii*) *For every* $s \geq 2$,

$$\sum_{k=1}^{T} |u(k)|^s \geq T^{(2-s)/2} \|u\|_e^s \text{ for all } u \in E_T.$$

(*iii*) *For all* $u \in E_T$,

$$\sum_{k=1}^{T} |\triangle u(k-1)|^{p(k-1)} \leq T\left(2^{p^+}\|u\|_e^{p^+} + 1\right).$$

**Lemma 5.2.** *Assume that* $(A1)$, $(A3)$ *and* $(F7)$ *hold with* $p^+ < r^-$. *Then, for any* $\lambda > 0$, *the functional* $J_T$ *given by* (5.2) *is anticoercive on* $E_T$. *Moreover, for* $r^- = p^+$, *there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (\lambda^*, \infty)$, *the functional* $J_T$ *is anticoercive on* $E_T$.

Proof. As in the proof of Theorem 3.1, we see by $(F7)$ that

$$F(k,t) \geq \frac{\beta(k)}{r(k)}|t|^{r(k)}, \tag{5.3}$$

for all $k \in \mathbb{Z}$ and $t \in \mathbb{R}$. By assumptions $(A1)$, $(A3)$, relation (5.3) and Lemma 5.1-($i$)-($ii$)-($iii$), we see that

$$\begin{aligned}
J_T(u) &\leq m_1 \int_0^{\zeta[u]} d\xi - \lambda\beta^- \sum_{k=1}^{T} \frac{|u(k)|^{r(k)}}{r(k)} \\
&\leq m_1 \left( a_1^+ \sum_{k=1}^{T} |\triangle u(k-1)| + \frac{a_2}{p^-} \sum_{k=1}^{T} |\triangle u(k-1)|^{p(k-1)} + \frac{q^+}{p^-} \sum_{k=1}^{T} |u(k)|^{p(k)} \right) \\
&- \frac{\lambda\beta^-}{r^+} \sum_{k=1}^{T} |u(k)|^{r(k)} \\
&\leq \left(2^{p^+}\left(a_1^+ + \frac{a_2}{p^-}\right) + \frac{q^+}{p^-}\right) m_1 T \|u\|_e^{p^+} - \frac{\lambda\beta^-}{r^+} T^{(2-r^-)/2} \|u\|_e^{r^-} + \left(2a_1^+ + \frac{a_2}{p^-}\right) m_1 T.
\end{aligned}$$

Hence, $J_T(u) \to -\infty$ as $\|u\|_e \to \infty$, since $r^- > p^+$. Therefore $J_T$ is anticoercive on $E_T$.
Let us now assume that $r^- = p^+$. As before, we obtain from the above estimation that

$$J_T(u) \leq \left(\left(2^{p^+}\left(a_1^+ + \frac{a_2}{p^-}\right) + \frac{q^+}{p^-}\right) m_1 T - \frac{\lambda\beta^-}{r^+} T^{(2-p^+)/2}\right) \|u\|_e^{p^+} + \left(2a_1^+ + \frac{a_2}{p^-}\right) m_1 T.$$

Therefore, if we choose

$$\lambda^* = \frac{\left(2^{p^+}\left(a_1^+ + \frac{a_2}{p^-}\right) + \frac{q^+}{p^-}\right) r^+ m_1 T^{p^+/2}}{\beta^-},$$

then, for any $\lambda \in (\lambda^*, \infty)$, the functional $J_T$ is anticoercive on $E_T$.

**Theorem 5.3.** *Suppose that the hypotheses* $(A1)$-$(A4)$, $(F1)$-$(F3)$ *and* $(F7)$-$(F9)$ *hold and that* $p^+ < r^-$. *Then, for any* $\lambda \in \left(\frac{r^+ m_1 q^+ T^{r^-/2}}{p^- \beta^-}, \frac{m_0 \min\{1,c\} D^{p^+}}{p^+ \epsilon T}\right)$, *problem* (5.1) *has at least three* $T$*-periodic solutions, at least two of which are nontrivial. When* $r^- = p^+$, *there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (\lambda^*, \infty)$, *problem* (5.1) *has at least three* $T$*-periodic solutions, at least two of which are nontrivial.*

Proof. By Lemma 2.1, we obtain

$$F(k,t) - F(k,0) = \langle w, t\rangle,$$

for some $w \in \partial F(k, \theta t)$ and $\theta \in (0,1)$. The above equality and condition $(F1)$ imply that

$$F(k,t) = \langle w, t\rangle,$$

for some $w \in \partial F(k, \theta t)$ and $\theta \in (0,1)$. Fix $0 < \epsilon < \dfrac{m_0 \min\{1,c\} D^{p^+}}{p^+ \lambda T}$. By $(F8)$ and $(F9)(i)$, there exists $\rho \in (0,d)$ such that

$$|F(k,t)| \le |\langle w, t\rangle| \le \epsilon |t|^{r^-}, \tag{5.4}$$

for all $k \in \mathbb{Z}$ and all $|t| \le \rho$.
Let $u \in Y_a$ with $\|u\|_e \le \rho$. Then, $|u(k)| \le \rho$ for any $k \in \mathbb{Z}$, so by $(A2)$-$(A4)$, (5.4) and Lemma 5.1$(i)$, one has

$$\begin{aligned} J_T(u) &\ge m_0 \int_0^{\zeta[u]} d\xi - \lambda \varepsilon \sum_{k=1}^{T} |u(k)|^{r^-} \\ &\ge \frac{m_0 \min\{1,c\}}{p^+} \|u\|_{p^+}^{p^+} - \lambda \epsilon T \|u\|_e^{r^-}. \end{aligned}$$

Since on $Y_a$, all norms are equivalent, then there exists $D > 0$ such that

$$\|u\|_{p^+} \ge D \|u\|_e, \text{ for all } u \in Y_a.$$

So,

$$\begin{aligned} J_T(u) &\ge \frac{m_0 \min\{1,c\}}{p^+} D^{p^+} \|u\|_e^{p^+} - \lambda \epsilon T \|u\|_e^{r^-} \\ &\ge \left(\frac{m_0 \min\{1,c\}}{p^+} D^{p^+} - \lambda \epsilon T\right) \|u\|_e^{r^-}, \text{ for } u \in Y_a \text{ with } \|u\|_e \le \rho. \end{aligned}$$

For $u \in W_a$ such that $\|u\|_e \le \rho$, $|u(k)| \le d$ for any $k \in \mathbb{Z}$, thus, by $(A1)$, $(A3)$ and (5.3), we get

$$\begin{aligned} J_T(u) &\le m_1 \int_0^{\zeta[u]} d\xi - \lambda \beta^- \sum_{k=1}^{T} \frac{|u(k)|^{r(k)}}{r(k)} \\ &\le m_1 \left( a_1^+ \sum_{k=1}^{T} |\Delta u(k-1)| + \frac{a_2}{p^-} \sum_{k=1}^{T} |\Delta u(k-1)|^{p(k-1)} + \frac{q^+}{p^-} \sum_{k=1}^{T} |u(k)|^{p(k)} \right) \end{aligned}$$

$$-\frac{\lambda\beta^-}{r^+}\sum_{k=1}^{T}|u(k)|^{r(k)}. \tag{5.5}$$

Moreover, since for any $(k,u)\in\mathbb{Z}\times W_a$ one has $\triangle u(k-1)=0$, then by (5.5) and Lemma 5.1-$(i)$-$(ii)$, it follows that

$$\begin{aligned} J_T(u) &\leq \frac{m_1q^+T}{p^-}\|u\|_e^{p^+}-\frac{\lambda\beta^-}{r^+}T^{(2-r^-)/2}\|u\|_e^{r^-} \\ &\leq \left(\frac{m_1q^+T}{p^-}-\frac{\lambda\beta^-}{r^+}T^{(2-r^-)/2}\right)\|u\|_e^{r^-},\ \text{ for } u\in W_a \text{ with } \|u\|_e\leq\rho. \end{aligned}$$

This implies that $J_T$ has a local linking at 0 with respect to $E_T=Y_a\oplus W_a$.
If $\inf\limits_{u\in E_T}J_T(u)\geq 0$, then $J_T(v)=\inf\limits_{v\in W_a}J_T(v)=0$, for all $v\in W_a$ with $\|v\|_e\leq\rho$. Thus, $v\in W_a$ with $\|v\|_e\leq\rho$ are solutions of problem (5.1). On the other hand, if $\inf\limits_{u\in E_T}J_T(u)<0$, then 0 is not a minimizer of $J_T$.
Let us put $\psi_T=-J_T$. Note that by Lemma 5.2, we see that $\psi_T$ is coercive, therefore $\psi_T$ satisfies the (PS)-condition. Hence, $\psi_T$ is bounded below. Now, we note that the functional $\psi_T$ is coercive and continuous. Consequently, it has a minimizer. By using $(F9)(ii)$, we obtain

$$\inf_{u\in E_T}\psi_T(u)=\inf_{u\in E_T}(-J_T(u))=-\sup_{u\in E_T}J_T(u)<0.$$

Thus, $\psi_T$ has at least three critical points, and at least two of them are non-zero critical points. Thus, by Lemma 2.6, our problem (5.1) has at least three $T$-periodic solutions, at least two of which are nontrivial.

In the case $r^-=p^+$, the proof is similar to $r^->p^+$, so we omit it.

## 6. NON-VARIATIONAL PROBLEM

In this section, we deal with the existence of nontrivial solutions for the following Dirichlet boundary value problem.

$$\begin{cases} -M(\zeta[u])[\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1))-q(k)|u(k)|^{p(k)-2}u(k)] \\ \quad\in\partial F_1(k,u(k))-\partial F_2(k,u(k)),\ k\in\mathbb{Z}(1,T), \\ u(0)=u(T+1)=0, \end{cases} \tag{6.1}$$

where $F_1$ satisfy all the assumptions in Theorem 4.1 and $F_2$ satisfies the following assumptions.

$(F10)$ The function $k\mapsto F_2(k,t)$ is $T$-periodic for all $t\in\mathbb{R}$ and $F_2(k,0)=0$.
$(F11)$ The function $t\mapsto F_2(k,t)$ is locally Lipschitz for all $k\in\mathbb{Z}$.
$(F12)$ There exists a function $r(\cdot):\mathbb{Z}(1,T)\to(1,\infty)$ such that

$$|w|\leq q(k)|t|^{r(k)-1},\ \text{ for all } (k,t)\in\mathbb{Z}(1,T)\times\mathbb{R} \text{ and } w\in\partial F_2(k,t),$$

where $q(\cdot):\mathbb{Z}(1,T)\to(1,\infty)$ and $1<r(k)\leq r^+<p^-$.

$(F13)$ There exist $m_0, m_1, t_2, \theta, \nu \in (0,\infty)$ which satisfy $\theta > \nu > \dfrac{m_1}{m_0} p^+$ such that

$$F_2^\circ(k,t)t \le \nu F_2(k,t), \text{ for all } k \in \mathbb{Z}(1,T) \text{ with } |t| > t_2.$$

Let $\Psi_1, \Psi_2 : E \to \mathbb{R}$ be two functionals defined by

$$\Psi_1(u) = \sum_{k=1}^{T} F_1(k, u(k))$$

and

$$\Psi_2(u) = \sum_{k=1}^{T} F_2(k, u(k)).$$

Then, we define the energy functional $I : E \to \mathbb{R}$ corresponding to (6.1) by

$$I(u) = \Phi(u) - \Psi_1(u) + \Psi_2(u), \tag{6.2}$$

with $\Phi$ introduced in (2.7).
Arguing as in the proof of Lemma 2.5, we conclude that $I$ is the nonsmooth Lipschitz energy functional corresponding to problem (6.1).
One has the following result.

**Theorem 6.1.** *Assume that* $(A1)$-$(A4)$, $(F0)$, $(F4)$-$(F6)$ *and* $(F10)$-$(F13)$ *hold. Then, for any* $\lambda > 0$, *problem* (6.1) *has at least one nontrivial solution.*

Proof. Firstly, we show that $I$ satisfies the nonsmooth (PS)-condition. Suppose that $\{u_n\}_{n\ge1} \subseteq E$ such that

$$|I(u_n)| \le M_1 \text{ and } m(u_n) \to 0 \text{ as } n \to \infty, \tag{6.3}$$

where $M_1 > 0$ is a constant. As before, we choose $u_n^* \in \partial I(u_n)$ such that $m(u_n) = \|u_n^*\|$ for $n \ge 1$. Moreover, since $\|u_n^*\| \to 0$, one has

$$|\langle u_n^*, u_n\rangle| \le \varepsilon_n, \varepsilon_n \downarrow 0. \tag{6.4}$$

Note that $u_n^* = \Phi'(u_n) - w_n^1 + w_n^2$ for $n \ge 1$, where $\Phi' : E \to E^*$ as in the proof of Theorem 4.1 and $w_n^1 \in \partial F_1(k, u_n)$, $w_n^2 \in \partial F_2(k, u_n)$. Thus, by (6.2)-(6.4), $(A3)$-$(A4)$, $(F4)$, $(F13)$, we get

$$\begin{aligned}
M_1 + \frac{\varepsilon_n}{\theta} &\ge I(u_n) - \frac{1}{\theta}\langle u_n^*, u_n\rangle \\
&\ge \left(m_0 - \frac{m_1 p^+}{\theta}\right)\zeta[u_n] - \sum_{k=1}^{T} F_1(k, u_n(k)) + \sum_{k=1}^{T} F_2(k, u_n(k)) \\
&+ \frac{1}{\theta}\sum_{k=1}^{T}\left[w_n^1(k)u_n(k) - w_n^2(k)u_n(k)\right] \\
&\ge \left(m_0 - \frac{m_1 p^+}{\theta}\right)\zeta[u_n] + \frac{1}{\theta}\sum_{k=1}^{T}\left[F_1^\circ(k, u_n(k))u_n(k) - \theta F_1(k, u_n(k))\right]
\end{aligned}$$

$$+\frac{1}{\theta}\sum_{k=1}^{T}[\theta F_2(k,u_n(k)) - F_2^{\circ}(k,u_n(k))u_n(k)]$$
$$\geq \left(m_0 - \frac{m_1 p^+}{\theta}\right)\zeta[u_n].$$

For $n$ large enough, we may assume that $\|u_n\| > 1$. From $(A2)$, $(A4)$ and Lemma 2.7$(a)$, we obtain

$$M_1 + \frac{\varepsilon_n}{\theta} \geq \left(m_0 - \frac{m_1 p^+}{\theta}\right)\frac{\min\{1,c\}}{p^+}(\|u_n\|^{p^-} - (1+q^+)(T+1)).$$

Since $\theta > \dfrac{m_1}{m_0}p^+$ and $p^- > 1$, we see that $\{u_n\}$ is bounded in $E$. Hence, by passing to a subsequence, we may assume that $u_n \rightharpoonup u$ weakly in $E$. Next, we will prove that $u_n \to u$ in $E$ as $n \to \infty$. According to (6.4), one has

$$\langle \Phi'(u_n), u_n - u\rangle - \sum_{k=1}^{T} w_n^1(k)(u_n(k)-u(k)) + \sum_{k=1}^{T} w_n^2(k)(u_n(k)-u(k)) \leq \varepsilon_n \ \text{ for } \ n \geq 1.$$

On the other hand,

$$\sum_{k=1}^{T} w_n^1(k)(u_n(k)-u(k)) \to 0, \quad \sum_{k=1}^{T} w_n^2(k)(u_n(k)-u(k)) \to 0,$$

as $n \to \infty$, since $\{u_n - u\}$ in $E$ and $\{w_n^1\}_{n\geq 1}$, $\{w_n^2\}_{n\geq 1}$ in $E^*$ are bounded. Therefore,

$$\limsup_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle \leq 0.$$

Then,

$$\limsup_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0.$$

Using Lemma 2.8, one has $u_n \to u$ as $n \to \infty$. Thus, $I$ satisfies the nonsmooth (PS)-condition.

Now, we will show that $I$ satisfies the nonsmooth mountain pass theorem. By conditions $(F0)$ and $(F5)$, for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|F_1(k,t)| \leq \varepsilon |t|^{p^+} + c_\varepsilon q(k)|t|^{r(k)} \ \text{ for all } \ k \in \mathbb{Z}(1,T) \ \text{ and } \ t \in \mathbb{R}.$$

Let $u \in E$ with $\|u\| < 1$, the above inequality, the conditions $(A2)$-$(A4)$, Lemma 2.7$(b)$ and relation (2.6) imply that

$$\begin{aligned}
I(u) &\geq \widehat{M}(\zeta[u]) - \sum_{k=1}^{T} F_1(k,u(k)) \\
&\geq \frac{2^{\frac{p^- - p^+}{p^-}} m_0 \min\{1,c\}}{p^+ K^{p^+}}\|u\|^{p^+} - \sum_{k=1}^{T}\left(\varepsilon|u(k)|^{p^+} + c_\varepsilon q(k)|u(k)|^{r(k)}\right) \\
&\geq \frac{2^{\frac{p^- - p^+}{p^-}} m_0 \min\{1,c\}}{p^+ K^{p^+}}\|u\|^{p^+} - \varepsilon\kappa^{p^+}T\|u\|^{p^+} - c_\varepsilon q^+\kappa^{r^-}T\|u\|^{r^-}.
\end{aligned}$$

Let $\varepsilon > 0$ be small enough such that $0 < \varepsilon\kappa^{p^+}T < 2^{\frac{p^- - p^+}{p^-}} m_0 \min\{1, c\}/(2p^+K^{p^+})$. Therefore, we obtain

$$I(u) \geq \frac{m_0 \min\{1, c\}}{2^{\frac{p^+}{p^-}} p^+ K^{p^+}} \|u\|^{p^+} - c(\lambda, \varepsilon) q^+ \kappa^{r^-} T \|u\|^{r^-}.$$

Since $r^- > p^+$, there exist $R > 0$ small enough and $\varrho > 0$ such that $I(u) \geq \varrho > 0$ as $\|u\| = R$.

As in [51], we see by $(F4)$ and $(F13)$ that

$$F_1(k, tu) \geq t^\theta F_1(k, u), \text{ for all } k \in \mathbb{Z}(1, T), \ u \in \mathbb{R} \tag{6.5}$$

and

$$F_2(k, tu) \leq t^\nu F_2(k, u), \text{ for all } k \in \mathbb{Z}(1, T), \ u \in \mathbb{R}. \tag{6.6}$$

Then, it follows from $(A3)$, (6.2), (6.5) and (6.6) that for $v \in E \setminus \{0\}$ and $t > 1$,

$$\begin{aligned} I(tv) &= \widehat{M}(\zeta[tv]) + \sum_{k=1}^{T} F_2(k, tv(k)) - \sum_{k=1}^{T} F_1(k, tv(k)) \\ &\leq m_1 t^{\frac{m_1}{m_0}p^+}(\zeta[v]) + t^\nu \sum_{k=1}^{T} F_2(k, v(k)) - t^\theta \sum_{k=1}^{T} F_1(k, v(k)). \end{aligned}$$

Since $\theta > \nu > \dfrac{m_1}{m_0} p^+$, we see that $I(tv) \to -\infty$ as $t \to \infty$. Hence, we can choose $t_0 > 1$ large enough such that $e = t_0 v$ and $I(e) < 0$.

Thus, all the conditions of Lemma 2.3 are satisfied. Consequently, problem (6.1) has at least one nontrivial solution.

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