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OPTIMAL CONTROL OF IMPULSIVE SYSTEMS

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ABSTRACT. Necessary conditions are derived for impulsive control problems with constraints. The impulsive constraints are motivated by applications from epidemic and production control models. We consider optimal control of SIR epidemic models when susceptible and infected people are treated by vaccination or quarantine and medicinal help. The vaccination and certain medical treatments may be done at particular instants of scheduled time and quarantine and other treatments may be done continuously.

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1. Introduction

Mathematical aspects of impulsive hybrid control systems have been considered by engineers and mathematicians ([3],[4],[12], [25],[26], [28]).
In applications impulsive control problems have been useful in engineering and in finance, production control and inventory management, epidemic modeling. In production planning ([9], [11],[22], [24]), a decision maker may have to decide the proper quantity of products being produced at different times with the objective of maximizing profit over a planning horizon. The goals that a decision maker has to accomplish are generally complex and involve conflicting objectives. The decision maker must meet demands while adhering to industry requirement needs, capabilities, limitations, and restrictions. Depending on the particular application an appropriate model may be discrete or continuous time optimization problem.

In ([22]) a production-planning model conducive to optimization is developed and used with the preference-based optimization method: linear physical programming, multiobjective programming. In ([11]) a continuous-time aggregate production-planning is considered where the objective is to determine the total production-planning cost, which involves various sets of costs like production cost, subcontracting

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cost, over-time cost, hiring cost, firing cost, and inventory cost.

In epidemic modeling formulation of strategies to control or avoid the spread of epidemics are key components of design of public health policy. Some of the strategies to control or block the spread of epidemics consist of health education campaigns, contact tracing and screening and avoidance of contact, strategically timed mass vaccination, medical treatment and/or quarantine for those that are already infected. All these strategies have cost associated with them. Appropriate and timely strategies control cost ([5], [8], [10], [15], [17], [30]).

In this paper we start by considering a general impulsive control system. Keeping in mind application problems we will consider the control parameters lying in a closed interval between zero and an upper bound. This is natural due to the fact that resources are limited, be it financial, technical, etc. In epidemic modeling the allocation of resources depends on the size of the population, the stage of the epidemic, and probable length of epidemic time horizon ([6], [8], [17], [15]).

In general a model depends on population homogeneity and migration. The model could consider multiple groups, and the level of interaction between groups. The level of available resources also depends on the size of population. The epidemic model also significantly differs depending on the particular epidemic under consideration. Influenza model is quite different from HIV model ([6], [8], [10], [14], [15], [18], [19]).

Thus, control models should possibly consider multiple groups, the particular epidemic, time horizon, control objectives, population size, state of the epidemic and available resources ([6], [8], [10], [14], [15], [17], [30]). In this paper we consider simple models ([16], [17]) where different controls are applied between distinct time intervals in the planning/epidemic horizon. We start with a general mathematical model where the SIR models of interest are particular cases. We characterize the optimal controls for the general model. We apply the results to the concrete models where numerical results are presented.

2. **Control Problem**

Let $0 = t_0 < t_1 < \cdots < t_{i-1} < t_i < \cdots < t_{N-1} < t_N = t_f$ be instants in the time horizon $[t_0, t_f]$ where the dynamics of our control system abruptly changes. The general problem we consider has the form

$$\min \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f_i^0(\phi_i(t), u_i(t), t))dt + T(\phi_N(t_N), t_N)$$

subject to

$$(2.1)\qquad \begin{aligned} \phi_i'(t) &= f_i(\phi_i(t), u_i(t), t), \qquad t_{i-1} < t < t_i \\ G_i(\phi_i(t_i^-), \phi_{i+1}(t_i^+)) &= 0 \quad , \quad i = 1, \cdots, N-1 \\ G_0(\phi_1(t_0)) &= 0 \\ H_i(\phi_i(t_i^-), \phi_i(t_{i-1}^+)) &\leq 0 \quad , \quad i = 1, \cdots, N \end{aligned}$$

Assumption 2.1.

(i) The function

$$f_i : \mathcal{X}_i \times \mathcal{U}_i \times [t_0, t_f] \longrightarrow \mathcal{R}^n$$

where $\mathcal{X}_i$ an open interval in $\mathcal{R}^n$, $\mathcal{U}_i$ an open interval in $\mathcal{R}^m$ is differentiable in x and continuous in x and u, for each fixed t. It is measurable in t for each fixed $(x, u) \in \mathcal{X}_i \times \mathcal{U}_i$

(ii) For each compact set $\Gamma_i \subset \mathcal{X}_i \times \mathcal{U}_i$ there exists a function $\Lambda \in L_2(t_0, t_f)$ such that $\forall(x, u, t) \in \Gamma_i \times [t_0, t_f]$

$$|f_i(x, u, t)| + |f_{i1}(x, u, t)| \leq \Lambda(t)$$

where f_{i1} denotes the matrix of partial derivatives of f_i with respect to x. Here, and in what follows, $|\cdot|$ denotes the Euclidean norm of the vector or matrix in question. The function f_i^0 satisfies the same properties in (i) and (ii) stated for f_i.

(iii) The functions $G_i : \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}$, $H_i : \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}$, $G_0 : \mathcal{R}^n \to \mathcal{R}$ are continuously differentiable.

(iv) To guarantee that all admissible trajectories lie in a fixed compact set $\mathcal{X}_i' \subset \mathcal{X}_i$ we require that $\forall(x, u) \in \mathcal{X}_i \times \mathcal{U}_i$ we have

$$|\langle x, f_i(x, u, t)\rangle| \leq W(t)\left(1 + |x|^2\right), \quad W \in L_1(t_0, t_f)$$

(v) We assume that $u_i(t) \in \Omega_i$ a.e., where Ω_i is a fixed compact subset of $\mathcal{U}_i$

Remark 2.1. For the concept of relaxed control see ([7]).

3. Admissible Pairs and Functional

We say that $((\phi_1, \cdots, \phi_N), (u_1, \cdots, u_N))$ is an admissible pair if it satisfies (2.1). We assume that the set of admissible pairs is not empty. In this paper we use relaxed controls and write $\langle(\phi_1, \cdots, \phi_N), (v_1, \cdots, v_N)\rangle$ for admissible pair.

Consider the functional

$$
\begin{aligned}
F_K((\phi_1,\cdots,\phi_N),(v_1,\cdots,v_N)) \quad = \quad & \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} f_i^0(\phi_i(t),v_i(t),t)dt + T(\phi_N(t_N),t_N) \\
& + \sum_{i=1}^{N} \| \phi_i^{'} - \phi_{i0}^{'} \|^2 + \sum_{i=1}^{N} |\phi_i(0)-\phi_{i0}(0)|^2 + \epsilon \sum_{i=1}^{N} ||v_i - v_{i0}||_L \\
& + K\sum_{i=1}^{N-1} G_i^2(\phi_i(t_i^-),\phi_{i+1}(t_i^+)) + KG_0^2(\phi_i(t_0)) \\
& + K\sum_{i=1}^{N} \omega(H_i((\phi_i(t_i^-),\phi_i(t_{i+1}^+))) \\
& + K\sum_{i=1}^{N} ||\phi_i{'} - f_i(\phi_i(t),v_i(t),t)||^2 \qquad (3.1)
\end{aligned}
$$

where $((\phi_{10},\cdots,\phi_{N0}),(v_{10},\cdots,v_{N0}))$ is an optimal pair for the above minimization problem which is fixed from now on, and

$$||v_i - v_{i0}||_L = ess - sup\{|v_i(t) - v_{i0}(t)|(\Omega),\ t_{i-1} \le t \le t_i\},$$

$$||\cdot|| = L_2 - norm, \quad \omega \in C^1(\mathcal{R}),$$

$$\omega(\xi) = 0 \text{ if } \quad \xi \le 0 \quad \text{and} \quad \omega(\xi) > 0 \quad \text{if} \quad \xi > 0$$

$$||\phi_i^{'} - \phi_{i0}^{'}||^2 = ||\phi_i^{'} - \phi_{i0}^{'}||^2_{L_2(t_{i-1},t_i)}$$

Assume that

$$\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} f_i^0(\phi_{i0}(t),v_{i0}(t),t)dt + T(\phi_N(t_N),t_N) = 0, \quad i = 1,\cdots,N$$

Then,

$$F_K(\phi_{10},\cdots,\phi_{N0},v_{10},\cdots,v_{N0}) = 0$$

Notation 3.1. Let

$$\mathcal{B} = \{(\phi_1,...,\phi_N),(v_1,...,v_N)|\phi_i \ absolutely\ continuous\ \phi_i{'} \in [L_2(t_{i-1},t_i)]^n,$$

$$v_i\ relaxed\ control, G_0(\phi_1(t_0)) = 0,\ G_j(\phi_j(t_j^-),\phi_{j+1}(t_j^+)) = 0, H_i(\phi_i(t_i^-),\phi_i(t_{i-1}^+)) \le 0$$

$$H_i(\phi_i(t_i^-),\phi_i(t_{i-1}^+)) \le 0,\ G_0(\phi_1(t_0)) = 0, \}$$

Choose $\epsilon_1 > 0$ so that

$$(||\phi_i^{'} - \phi_{i0}^{'}|| \le \epsilon_1, |\phi_i(0) - \phi_{i0}(0)| \le \epsilon_1) \Rightarrow (\phi_i(t) \in \mathcal{X}_i^{'}, \quad t_{i-1} \le t \le t_i)$$

where $\mathcal{X}_i^{'}$ is in Assumptions 2.1.

Lemma 3.1. *For any* $0 < \epsilon \leq \epsilon_1$ $\exists K(\epsilon) > 0$, *such that* $F_{K(\epsilon)}(\phi_1, \cdots, \phi_N, v_1, \cdots, v_N) > 0$, *if* $((\phi_1, \cdots, \phi_N), (v_1, \cdots, v_N)) \in \mathcal{B}$ *and any of the following inequalities is an equality:*
$|\phi_i(0) - \phi_{i0}(0)| \leq \epsilon, \quad ||\phi_i^{'} - \phi_{i0}^{'}|| \leq \epsilon, \quad ||v_i - v_{i0}||_L \leq \epsilon$

Proof. Suppose the lemma were false. Then, for any $0 < \epsilon \leq \epsilon_1$ $\exists((\phi_1^j, \cdots, \phi_N^j), (v_1^j, \cdots, v_N^j)) \in \mathcal{B}$, $j = 1, \cdots$, and $K_1, \cdots, K_j, \cdots K_j \to \infty$ where $((\phi_1^j, \cdots, \phi_N^j), (v_1^j, \cdots, v_N^j))$ satisfies one of the equalities such that $F_{K_j}(\phi_1^j, \cdots, \phi_N^j, v_1^j, \cdots, v_N^j) \leq 0$. There exists $j_1 < j_2 < \cdots < j_k < \cdots$ and $((\phi_1^*, \cdots, \phi_N^*), (v_1^*, \cdots, v_N^*))$ such that ${\phi_i^{j_k}}' \to {\phi_i^*}'$ weakly in $[L_2(t_{i-1}, t_i)]^n$, $\phi_i^{j_k} \to \phi_i^*$ uniformly in $[t_{i-1}, t_i]$, $v_i^{j_k} \to v_i^*$ weak-star on $[t_{i-1}, t_i]$. It is clear that $((\phi_1^*, \cdots, \phi_N^*), (v_1^*, \cdots, v_N^*)) \in \mathcal{B}$ and

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f_i^0(\phi_i^*(t), v_{it}^*, t)dt + T(\phi_N(t_N), t_N) + \epsilon^2 \leq 0$$

We have arrived at a contradiction and the lemma is proved.

□

Notation 3.2. Let

$$\begin{aligned}\mathcal{B}(\epsilon) \quad = \quad & \{((\phi_1, \cdots, \phi_N), (v_1, \cdots, v_N)) \in \mathcal{B} : \ ||\phi_i^{'} - \phi_{i0}^{'}|| \leq \epsilon, \\ & |\phi_i(0) - \phi_{i0}(0)| \leq \epsilon, ||v_i - v_{i0}||_L \leq \epsilon\}\end{aligned}$$

Corollary 3.1. Let $K(\epsilon)$ as in Lemma 3.1, $0 < \epsilon \leq \epsilon_1$. Then, the functional $((\phi_1, \cdots, \phi_N), (v_1, \cdots, v_N)) \mapsto F_{K(\epsilon)}(\phi_1, \cdots, \phi_N, v_1, \cdots, v_N)$ attains its minimum on $\mathcal{B}$. Further, letting

$$\begin{aligned}& F_{K(\epsilon)}(\phi_{1\epsilon}, \cdots, \phi_{N\epsilon}, v_{1\epsilon}, \cdots, v_{N\epsilon}) \\ = \quad & \inf\{F_{K(\epsilon)}(\phi_1, \cdots, \phi_N, v_1, \cdots, v_N) | ((\phi_1, \cdots, \phi_N), (v_1, \cdots, v_N)) \in \mathcal{B}\}\end{aligned}$$

we have

$$\|\phi_{i\epsilon}^{'} - \phi_{i0}^{'}\| < \epsilon, \quad \|\phi_{i\epsilon}(0) - \phi_{i0}(0)\| < \epsilon, \quad ||v_{i\epsilon} - v_{i0}||_L < \epsilon, \quad i = 1, \cdots, N.$$

Let η be an absolutely continuous function with compact support in (t_{i-1}, t_i) and $\eta^{'} \in [L_2(t_{i-1}, t_i)]^n$. Then,

$$dF_K(\phi_{1\epsilon}, \cdots, \phi_{i\epsilon} + \theta\eta, \phi_{i+1,\epsilon}, \cdots, \phi_{N\epsilon}, v_{1\epsilon}, \cdots, v_{N\epsilon})/d\theta = 0. \tag{3.2}$$

From (3.2) we have

$$\begin{aligned}& \int_{t_{i-1}}^{t_i} f_i^0(\phi_{i\epsilon}(t), v_{i\epsilon}(t), t)\eta \, dt + \int_{t_{i-1}}^{t_i} 2(\phi_{i\epsilon}^{'} - \phi_{i0}^{'})\dot{\eta}^{'} \\ + \quad & K(\epsilon) \int_{t_{i-1}}^{t_i} 2(\phi_{i\epsilon}^{'} - f_{i\epsilon}(\phi_{i\epsilon}(t), v_{i\epsilon}(t), t))(\eta^{'} - f_{i1}\eta) \, dt = 0\end{aligned} \tag{3.3}$$

Setting

$$\psi_i(\epsilon;t) = 2(\phi_{i\epsilon}^{'} - \phi_{i0}^{'}) + 2K(\epsilon)(\phi_{i\epsilon}^{'} - f_i(\phi_{i\epsilon}(t), v_{i\epsilon t}, t)) \tag{3.4}$$

we have

$$\begin{aligned}\psi_i(\epsilon;t) = \psi_i(\epsilon;t_{i-1}) \quad &+ \quad \int_{t_{i-1}}^{t_i} \{f_{i1}^0(\phi_{i\epsilon}(s), v_{i\epsilon s}, s) - \psi_i(\epsilon;t)\cdot f_{i1}(\phi_{i\epsilon}(s), v_{i\epsilon s}, s)\\ &+ 2(\phi_{\epsilon}^{'} - \phi_0^{'})\cdot f_{i1}\}ds, \quad t_{i-1} \le t \le t_i\end{aligned} \tag{3.5}$$

Next, let $\delta > 0$ be such that $t_{i-2} < t_{i-1} - \delta < t_{i-1} < t_{i-1} + \delta < t_i$. Let η_δ be an absolutely continuous function with support in $(t_{i-1} - \delta, t_{i+1} + \delta)$, $\eta_\delta^{'} \in L_2(t_{i-1} - \delta, t_{i-1} + \delta)$ and $\eta(t_{i-1}) = 1, \eta \ge 0$. From (3.2) we obtain

$$\begin{aligned}&\int_{t_{i-1}}^{t_i} \{f_{i-1}^0\cdot\eta_\delta + \psi_{i-1}(\epsilon;t)\eta_\delta^{'} - \psi_{i-1}(\epsilon;t)\cdot f_{i-1,x}\eta_\delta + 2(\phi_{(i-1)\epsilon}^{'} - \phi_{(i-1)0}^{'}\cdot f_{i-1,x}\eta_\delta\}ds\\
+\ &\int_{t_{i-1}}^{t_i} \{f_{i-1}^0\cdot\eta_\delta + \psi_{i-1}(\epsilon;t)\eta_\delta^{'} - \psi_{i-1}(\epsilon;t)\cdot f_{i-1,x}\eta_\delta + 2(\phi_{(i-1)\epsilon}^{'} - \phi_{(i-1)0}^{'}\cdot f_{i-1,x}\eta_\delta\}ds\\
+\ &2K(\epsilon)G_{i-1,\epsilon}(\phi_{i-1,\epsilon}(t_{i-1}^-), \phi_{i\epsilon}(t_{i-1}^+)))G_{i-1,1}(\phi_{i-1,\epsilon}(t_{i-1}^-), \phi_{i\epsilon}(t_{i-1}^+))\\
+\ &2K(\epsilon)G_{i-1,1}(\phi_{i-1,\epsilon}(t_{i-1}^-), \phi_{i\epsilon}(t_{i-1}^+)))G_{i-1,2}(\phi_{i-1,\epsilon}(t_{i-1}^-), \phi_{i\epsilon}(t_{i-1}^+))\\
+\ &K(\epsilon)\,\omega^{'}(H_i(\phi_{i\epsilon}(t_i^-), \phi_{i\epsilon}(t_{i-1}^+))H_{i,2}(\phi_{i\epsilon}(t_i^-), \phi_{i\epsilon}(t_{i-1}^+)))\\
+\ &K(\epsilon)\,\omega^{'}(H_{i-1}(\phi_{i-1,\epsilon}(t_{i-1}^-), \phi_{i-1,\epsilon}(t_{i-2}^+))H_{i-1,1}(\phi_{i-1,\epsilon}(t_{i-1}^-), \phi_{i-1,\epsilon}(t_{i-2}^+))) = 0\end{aligned} \tag{3.6}$$

From (3.6) we infer that

$$\begin{aligned}&\psi_{i-1}(\epsilon;t_{i-1}) - \psi_i(\epsilon, t_{i-1})\\
+\ &2K(\epsilon)G_{i-1,\epsilon}(\phi_{i-1,\epsilon}(t_{i-1}^-), \phi_{i\epsilon}(t_{i-1}^+))\sum_{j=1}^{2} G_{i-1,j}(\phi_{i-1}(t_{i-1}^-), \phi_i(t_{i-1}^+))\\
+\ &K(\epsilon)\,\omega^{'}(H_i(\phi_{i\epsilon}(t_i^-), \phi_{i\epsilon}(t_{i-1}^+))H_{i,2}(\phi_{i\epsilon}(t_i^-), \phi_{i\epsilon}(t_{i-1}^+)))\\
+\ &K(\epsilon)\,\omega^{'}(H_{i-1}(\phi_{i-1,\epsilon}(t_{i-1}^-), \phi_{i-1,\epsilon}(t_{i-2}^+))H_{i-1,1}(\phi_{i-1,\epsilon}(t_{i-1}^-), \phi_{i-1,\epsilon}(t_{i-2}^+))) = 0\end{aligned} \tag{3.7}$$

Making similar variations at t_N and t_0 we obtain

$$\begin{aligned}&\psi_N(\epsilon;t_N) + T_x(\phi_{N_\epsilon}(t_N), t_N)\\
+\ &2K(\epsilon)G_{N-1}(\phi_{N-1,\epsilon}(t_{N-1}^-), \phi_{N\epsilon}(t_N^+))\cdot G_{N-1,2}(\phi_{N-1,\epsilon}(t_{N-1}^-), \phi_{N\epsilon}(t_N^+))\\
+\ &K(\epsilon)\omega^{'}(H_N(\phi_{N\epsilon}(t_N^-), \epsilon_{N\epsilon}(t_{N-1}^+))H_{N,1}(\phi_{N\epsilon}(t_N^-), \phi_{N\epsilon}(t_{N-1}^+))) = 0\end{aligned} \tag{3.8}$$

$$\begin{aligned}&-\psi_1(\epsilon;t_0)+2K(\epsilon)G_0(\phi_{1\epsilon}(t_0))G_{0,x}(\phi_{1\epsilon}(t_0))\\ &+K(\epsilon)\omega^{'}(H_1(\phi_{1\epsilon}(t_1^-),\phi_{1\epsilon}(t_0^+)(H_{i,1}(\phi_{1\epsilon}(t_1^-),\epsilon_{1\epsilon}(t_0^+)))=0\end{aligned}\tag{3.9}$$

For $0<\epsilon\le\epsilon_1$ and $0\le\theta\le 1$ let $v_i(\theta)=v_{i\epsilon}+\theta(v_i-v_\epsilon)$.

Since $\|v_{i\epsilon}-v_{i0}\|_L<\epsilon\ \exists\ \theta_0>0$ such that for $0<\theta\le\theta_0$ $\|v_{i\epsilon}(\theta)-v_{i0}\|_L<\epsilon$. Let

$$\rho_{i\epsilon}(\theta)=\|v_{i\epsilon}(\theta)-v_0\|_L.$$

we have

$$dF_{K(\epsilon)}(\phi_{1\epsilon},\cdots,\phi_{N\epsilon},v_{1\epsilon},\cdots,v_{i-1,\epsilon},v_i(\theta),v_{i+1,\epsilon},\cdots,v_{N\epsilon})/d\theta|_{\theta=0^+}\ge 0$$

leading to

$$\begin{aligned}&\int_{t_{i-1}}^{t_i}f_i^0(\phi_{i\epsilon}(t),v_{it},t)dt-\int_{t_{i-1}}^{t_i}\psi_(\epsilon;t)\cdot f_i(\phi_{i\epsilon}(t),v_{it},t)dt\\ +&\int_{t_{i-1}}^{t_i}2(\phi_{i\epsilon}^{'}-\phi_{i0}^{'})\cdot f_i(\phi_{i\epsilon}(t),v_{it},t)dt\\ \ge&\int_{t_{i-1}}^{t_i}f_i^0(\phi_{i\epsilon}(t),v_{i\epsilon t},t)dt-\int_{t_{i-1}}^{t_i}\psi_(\epsilon;t)\cdot f_i(\phi_{i\epsilon}(t),v_{i\epsilon t},t)dt\\ +&\int_{t_{i-1}}^{t_i}2(\phi_{i\epsilon}^{'}-\phi_{i0}^{'})\cdot f_i(\phi_{i\epsilon}(t),v_{i\epsilon t},t)dt-\epsilon\rho_{i\epsilon}^{'}(0^+)\end{aligned}\tag{3.10}$$

Let

$$\begin{aligned}M(\epsilon)&=1+2K(\epsilon)\sum_{i=1}^{N-1}|G_i(\phi_{i\epsilon}(t_i^-),\phi_{i+1,\epsilon}(t_i^+))|+2K(\epsilon)\ |G_0(\phi_{1\epsilon}(t_0))|\\ &+\ K(\epsilon)\sum_{i=1}^{N}|\omega^{'}(H_i(\phi_{i\epsilon}(t_i^-),\phi_{i\epsilon}(t_{i-1,\epsilon}^+)))|+\sum_{i=1}^{N}\|\psi_i(\epsilon,\cdot)\|_\infty\end{aligned}$$

In (3.4) – (3.10) we divide by $M(\epsilon)$ and take an appropriate subsequence of $\epsilon>0$ tending to zero to obtain the following equations where we have denoted the limit of $\{\psi_i(\epsilon;t)\ /\ M(\epsilon)\}_{\epsilon>0}$ as $\lambda_i(t)$.

$$\lambda_i(t)=\lambda_i(t_{i-1})+\int_{t_{i-1}}^{t_i}\{\lambda^0 f_{i1}^0(\phi_{i0}(s),v_{i0s},s)-\lambda_i(s)\cdot f_{i1}(\phi_{i0}(s),v_{i0s},s)\}ds\tag{3.11}$$

$$\begin{aligned}&\lambda_{i-1}(t_{i-1}^-)-\lambda_i(t_{i-1}^+)+\sum_{j=1}^{2}\beta_{i-1}G_{i-1,j}(\phi_{i-1,0}(t_{i-1}^-),\phi_{i0}(t_{i-1,0}^+))\\ +&\ \gamma_iH_{i,2}(\phi_{i0}(t_i^-),\phi_{i0}(t_{i-1}^+))+\xi_{i-1}H_{i-1,1}(\phi_{i-1,0}(t_{i-1}^-),\phi_{i-1,0}(t_{i-2}^+))=0\end{aligned}\tag{3.12}$$

$$\lambda_N(t_N) + \lambda^0 T_x(\phi_{N0}(t_N), t_N) + \beta_{N-1} G_{N,2}(\phi_{N-1,0}(t_{N-1}^-), \phi_{N0}(t_N^+)) + \gamma_N\ H_{N,1}(\phi_{N0}(t_N^-), \phi_{N0}(t_{N-1}^+)) = 0 \tag{3.13}$$

$$-\lambda_1(t_0) + \beta_0\ G_{0,x}(\phi_{10}(t_0)) + \gamma_1\ H_{1,1}(\phi_{10}(t_1^-), \phi_{10}(t_0^+) = 0 \tag{3.14}$$

$$\begin{aligned} & \lambda^0\ f_i^0(\phi_{i0}(t), v_{it}, t)) - \lambda_i(t) \cdot f_i(\phi_{i0}(t), v_{it}, t) \\ \geq\ & \lambda^0\ f_i^0(\phi_{i0}(t), v_{i0t}, t)) - \lambda_i(t) \cdot f_i(\phi_{i0}(t), v_{i0t}, t) \quad a.e. \end{aligned} \tag{3.15}$$

4. **SIR Model**

Let x represent the susceptible population, y the infected, and the z recovered. Following([17]) we consider the dynamics given by

$$\begin{aligned} \frac{dx}{dt} &= -\beta\frac{xy}{x+y},\ \beta > 0 \\ \frac{dy}{dt} &= \beta\frac{xy}{x+y} - \gamma y,\ \gamma > 0 \\ \frac{dz}{dt} &= \gamma y \\ x(0) &= x_0 \\ y(0) &= y_0 \end{aligned}$$

The effect of the vaccination is modeled by

$$\frac{d}{dt}x = -\beta\frac{xy}{x+y} - xu_1,\ 0 \leq a_1 \leq u_1 \leq b_1$$

The effect of medical treatment and/or quarantine is modeled by

$$\frac{d}{dt}y = \beta\frac{xy}{x+y} - \gamma y - yu_2,\ 0 \leq a_2 \leq u_2 \leq b_2$$

Finally the first cost is given by

$$J(u_1, u_2) = \int_0^{t_f} \{1/2y^2(t) + C_1u_1^2 + C_2u_2^2\}dt,\ C_1 > 0,\ C_2 > 0$$

Minimizing the cost is equivalent to decreasing the number of infected people and the cost. In this model we have two treatment strategies. The cost could be approached in a different way since decreasing the number of infected people and the cost are competing outcomes.

The impulsive version we consider is as follows. Consider $0 \leq t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t_f$. In the interval $[t_0, t_1]$ we consider the dynamics

$$\begin{aligned} \frac{dx}{dt} &= -\beta\frac{xy}{x+y} - xu_1,\ 0 \leq u_1 \leq 1 \\ \frac{dy}{dt} &= \beta\frac{xy}{x+y} - \gamma y - yv_1,\ 0 \leq u_2 \leq 1,\ \gamma > 0 \\ \frac{dz}{dt} &= \gamma y \\ x(0) &= x_0 \\ y(0) &= y_0 \end{aligned} \tag{4.1}$$

In the interval $[t_{i-1}, t_i],\ i = 2, 3, \cdots, n$ we consider we consider

$$\begin{aligned} \frac{dx_i}{dt} &= -\beta\frac{x_i y_i}{x_i+y_i} - x_i u_i,\ 0 \leq u_i \leq 1 \\ \frac{dy_i}{dt} &= \beta\frac{x_i y_i}{x_i+y_i} - \gamma y_i - y_i v_i,\ 0 \leq v_i \leq 1,\ \gamma > 0 \\ \frac{dz_i}{dt} &= \gamma y_i \\ x_i(t_i) &= (1-a_i)x_{i-1}(t_{i-1}) \\ y_i(t_i) &= y_{i-1}(t_{i-1}) - q_i \\ x_i(t_i) &\leq x_{i-1}(t_{i-1}) - p_i \end{aligned} \tag{4.2}$$

The i-th cost is given by

$$J(u_i, v_i) = \int_{t_{i-1}}^{t_i} \{1/2y_i^2(t) + C_1u_i^2 + C_2v_i^2\}dt,\ C_1 > 0,\ C_2 > 0,\ i = 1, 2, \cdots, n$$

We would like to minimize

$$\sum_{i=1}^{n} J(u_i, v_i).$$

5. **Numerical Computation**

In this section we take the case $t_0 = 0, t_1 = 1, t_2 = 2, t_3 = 3 = t_f$ and numerically solve the problems in (4.1 and 4.2) with the corresponding cost $\sum_{i=1}^{3} J(u_i, v_i)$. We initialize the controls $(u_i, v_i), i = 1, 2, 3$ and use the method of steepest descent employing the minimum principle above following (3.8-3.13). In (4.2) a_i fraction of susceptibles vaccinated as we go to the next time interval. Similarly, q_i represents number of infected people that are quarantined, and we wish to reduce the number of susceptible individuals by p_i as we go from time t_{i-1} to t_i. We remark that a change in (u_3, v_3) should only affect the final objective value, while a change in (u_2, v_2) should affect the objective functions in the second and final time intervals. The results of the numerical computation are presented in figures 6 through figure 8. In the example we simulated we started with a susceptible population of 900 and infected 100. At the beginning of time interval 1 and 2 we removed/quarantined 10 infected people. We note the number of infected people has been decreased to 30 and susceptible people decreased to about 300, in each case to 30 % of the original population. We note that the decrease is steady in each population from one time interval to the next. We also note the total cost steadily decreasing from one time interval to the next.

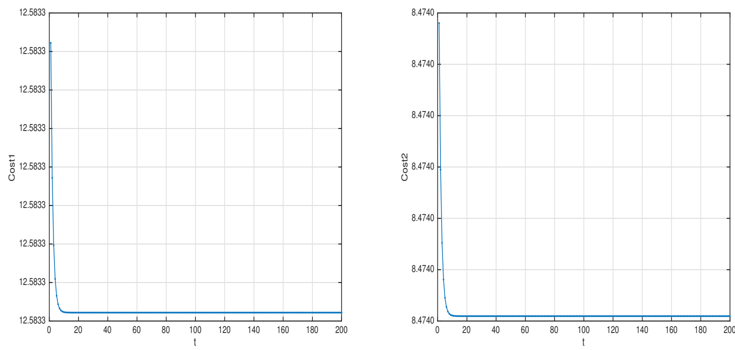

FIGURE 1. Total Cost in Time Interval 1 and 2.

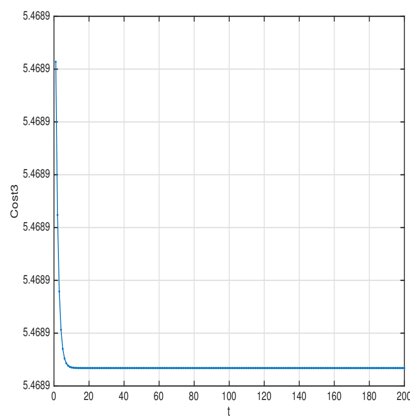

FIGURE 2. Total Cost in Time Interval 3.

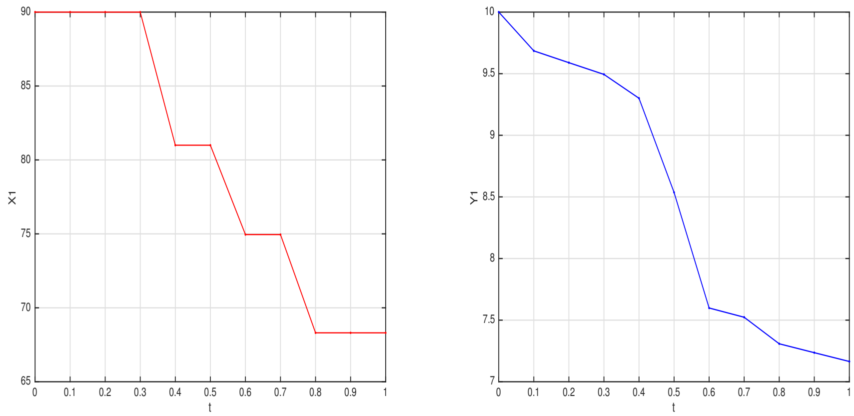

FIGURE 3. Susceptible & Infected Population in Time Interval 1.

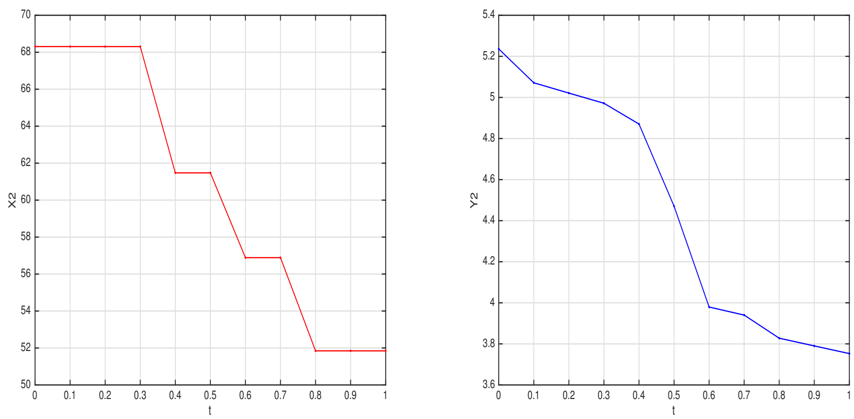

FIGURE 4. Susceptible & Infected Population in Time Interval 2.

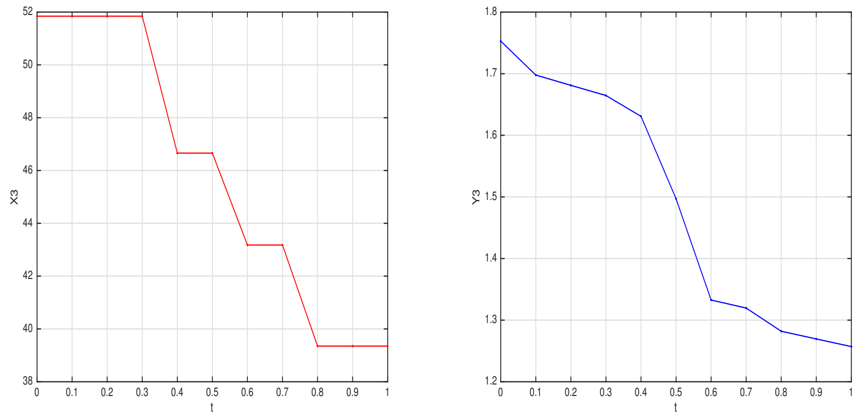

FIGURE 5. Susceptible & Infected Population in Time Interval 3.

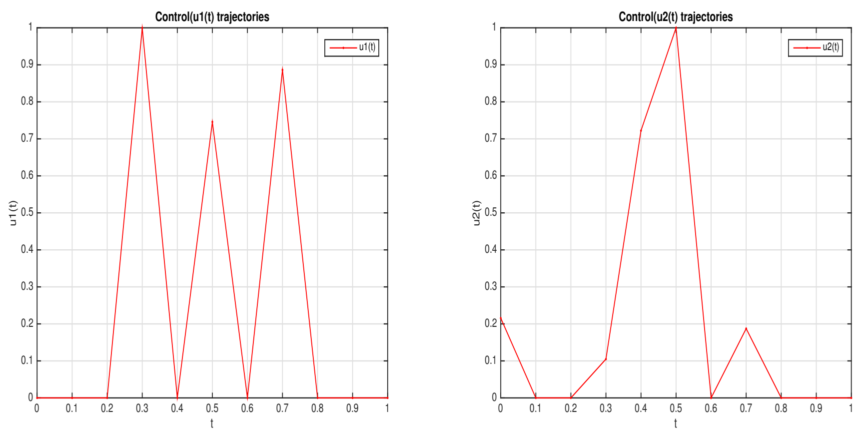

FIGURE 6. Susceptible Cost & Infected Cost in Time Interval 1.

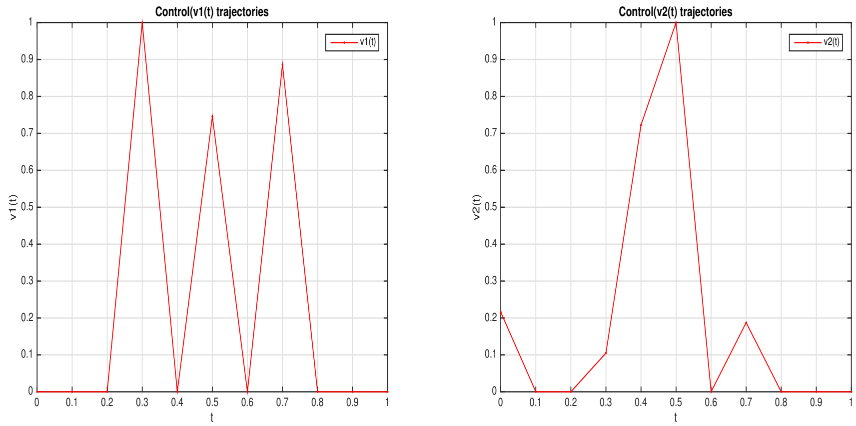

Figure 7. Susceptible Cost & Infected Cost in Time Interval 2.

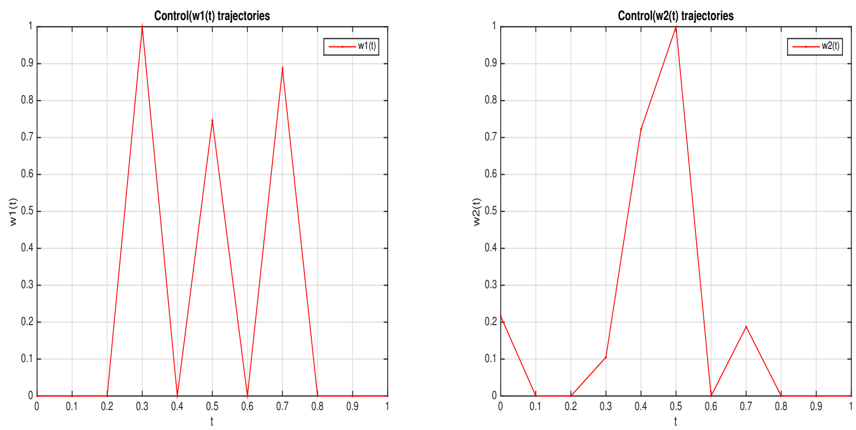

Figure 8. Susceptible Cost & Infected Cost in Time Interval 3.

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