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OSCILLATION CRITERIA FOR UNSTABLE SECOND-ORDER NONLINEAR DYNAMIC EQUATIONS WITH MIXED TYPE DEVIATING ARGUMENTS

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ABSTRACT. The goal of this work is to establish new criteria for the oscillation of all solutions of the second-order dynamic equation with mixed arguments

$$\left(r(\nu)y^{\Delta}(\nu)\right)^{\Delta} = q(\nu)f(y(\tau(\nu))).$$

The approach used employs the existence of an suitable auxiliary function and integral averaging principles. This makes it possible to improve many existing criteria known for such equations. We note that the results presented here are new even in the cases where $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. The results are illustrated by some examples.

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1. INTRODUCTION

In this work we are interested in the oscillatory behavior of all solutions of the second-order nonlinear dynamic equation

$$\left(r(\nu)y^{\Delta}(\nu)\right)^{\Delta} = q(\nu)f(y(\tau(\nu))) \tag{1.1}$$

on a time scale $\mathbb{T}$. Here, $\sup \mathbb{T} = \infty$ and $\nu \in [\nu_0, \infty)_{\mathbb{T}} = [\nu_0, \infty) \cap \mathbb{T}$ with $\nu_0 \in \mathbb{T}$. The notation and terminology for equations on time scales as found in Bohner and Peterson [9, 10] (also see Agarwal et al. [1, 2] and Georgiev [15]) will be used throughout this paper.

Equations of the form of (1.1) with $q > 0$ are often referred to as being of the unstable type or the Thomas-Fermi type; if $q < 0$, they are often said to be of the Emden-Fowler type.

For equation (1.1), we always assume that:

(H$_1$) $r \in \mathrm{C}^1_{rd}([\nu_0, \infty)_{\mathbb{T}}, \mathbb{R}_+)$ and $q \in \mathrm{C}_{rd}([\nu_0, \infty)_{\mathbb{T}}, \mathbb{R}_+)$ with $q(\nu) \not\equiv 0$;

(H$_2$) $\tau \in \mathrm{C}^1_{rd}([\nu_0, \infty)_{\mathbb{T}}, [\nu_0, \infty)_{\mathbb{T}})$ with $\lim_{\nu\to\infty} \tau(\nu) = \infty$;

(H$_3$) $R(\nu) = \int_{\nu_0}^{\nu} \frac{1}{r(s)}\Delta s \to \infty \quad \text{as} \quad \nu \to \infty$;

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(H_4) there exists $K > 0$ such that $\frac{f(u)}{u} \geq K$ for all $u \neq 0$.

By a *solution* of (1.1), we mean a function y that is rd-continuous on $[\nu_0, \infty)_{\mathbb{T}}$ and satisfies (1.1) on $[T_y, \infty)_{\mathbb{T}}$ for some $T_y \geq \nu_0$. We only consider those solutions y of (1.1) that are continuable in the sense that $\sup\{|y(\nu)| : \nu \geq T\} > 0$ for all $T > T_y \geq \nu_0$, and we tacitly assume that equation (1.1) has such solutions. Such a solution is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and *nonoscillatory* otherwise.

Definition 1.1. The argument $\tau(\nu)$ is said to be of mixed type if the delayed part of $\tau(\nu)$, defined as

$$D_\tau = \{\nu \in (\nu_0, \infty)_{\mathbb{T}} : \tau(\nu) < \nu\},$$

and the advanced part of $\tau(\nu)$, defined as

$$A_\tau = \{\nu \in (\nu_0, \infty)_{\mathbb{T}} : \tau(\nu) > \nu\},$$

are unbounded subsets of $[0, \infty)_{\mathbb{T}}$.

As observed in [3], there are natural mechanisms that occur over both a continuous and a discrete time period. This somewhat provides a motivation for the above definition. In the case where $\mathbb{T} = \mathbb{R}$, then $\sigma(\nu) = \nu$ and (1.1) takes the form of a differential equation

$$(r(\nu)y'(\nu))' = q(\nu)f(y(\tau(\nu))). \tag{1.2}$$

while if $\mathbb{T} = \mathbb{Z}$, then $\sigma(\nu) = \nu + 1$ and (1.1) becomes the difference equation

$$\Delta\left(r(\nu)\Delta y(\nu)\right) = q(\nu)f(y(\tau(\nu))). \tag{1.3}$$

If for some fixed $h > 0$, we have $\mathbb{T} = h\mathbb{Z} = \{nh : n \in \mathbb{Z}\}$, then $\sigma(\nu) = \nu + h$ and (1.1) becomes

$$\Delta_h\left(r(\nu)\Delta_h y(\nu)\right) = q(\nu)f(y(\tau(\nu))). \tag{1.4}$$

If for fixed $q > 1$, $\mathbb{T} = q^{\mathbb{Z}} = \{q^n : n \in \mathbb{N}_0\}$, then $\sigma(\nu) = q\nu$ and (1.1) takes the form

$$\Delta_q\left(r(\nu)\Delta_q f(y(\nu))\right) = q(\nu)f(y(\tau(\nu))). \tag{1.5}$$

Equations (1.2) and (1.3) have been studied by many researchers and there are a good deal of results on these equations when the deviating argument is either a delay ($\tau(\nu) \leq \nu$) or an advanced ($\tau(\nu) \geq \nu$) argument; see, for example, [11, 13, 25, 26, 27]. Kusano [22] was one of the first to consolidate these cases and use the terminology of deviating arguments of mixed type. We direct the reader to the publications [4, 5, 6, 7, 12] for generalizations and/or enhancements of the results in [22]. There appears to be relatively few results on the oscillation of equations (1.3), (1.4) or (1.5); see, for example, [14, 23]. With this in mind, our aim here is to investigate the more general equation (1.1). Part of the motivation for this study stems from a very nice paper by Baculíková and Džurina [7].

The reminder of this paper is organized as follows. Section 2 contains some observations about the solutions of (1.1) and their derivatives. In Subsections 2.1 and 2.2, we present our results on the non-existence of positive decreasing and positive increasing solutions, respectively, of equation (1.1). Section 3 contains our main theorems and some examples to illustrate them. In the final section of the paper, we summarize our findings and discuss future research directions.

2. **BASIC LEMMAS**

We assume that the inverse function τ^{-1} exists and that there is a function $\eta \in \mathrm{C}^1_{rd}([\nu_0,\infty)_{\mathbb{T}},\mathbb{T})$ such that

$$\eta(\eta(\nu)) = \tau^{-1}(\nu). \tag{2.1}$$

The following result is standard; see, for example, [1, Theorem 5].

Lemma 2.1. *If y is an eventually positive solution of Eq.* (1.1) *for $\nu \geq \nu_0$, then there exists $\nu_1 > \nu_0$ such that*

$$y \in C_0 \cup C_2,$$

where

$$C_0 = \{y : r(\nu)y^\Delta(\nu) < 0 \text{ and } (r(\nu)y^\Delta(\nu))^\Delta > 0 \text{ for } \nu \geq \nu_1\}$$

and

$$C_2 = \{y : r(\nu)y^\Delta(\nu) > 0 \text{ and } (r(\nu)y^\Delta(\nu))^\Delta > 0 \text{ for } \nu \geq \nu_1\}.$$

2.1. **Non-existence of C_0-type solutions.** In this subsection we present conditions that ensure that solutions of the type C_0 do not exist. To accomplish this, we assume the existence of a sequence

$$\{\nu_k\} \to \infty \text{ with } \nu_k \in D_\tau. \tag{2.2}$$

It is easy to see that $\eta(\nu_k) > \nu_k$.

For convenience of notation, we introduce the following functions:

$$R_1(\nu_k) = \int_{\nu_k}^{\eta(\nu_k)} [R(\sigma(x)) - R(\nu_k)]q(x)\Delta x,$$
$$R_2(\nu_k) = \int_{\eta(\nu_k)}^{\tau^{-1}(\nu_k)} [R(\sigma(x)) - R(\nu_k)]q(x)\Delta x,$$
$$R_3(\nu_k) = \int_{\tau^{-1}(\nu_k)}^{\tau^{-1}(\eta(\nu_k))} [R(\sigma(x)) - R(\nu_k)]q(x)\Delta x.$$

Lemma 2.2. *Assume that there is a sequence $\{\nu_k\}$ satisfying* (2.2) *and a function η with $\eta(\eta(\nu_k)) = \tau^{-1}(\nu_k)$ such that*

$$\limsup_{k\to\infty}\left[\frac{R_1(\nu_k)R_1(\eta^{-1}(\nu_k)) + R_1(\nu_k)R_3(\eta^{-1}(\nu_k))}{(1-KR_2(\nu_k))(1-KR_2(\eta^{-1}(\nu_k)))} + \frac{R_1(\eta(\nu_k))R_3(\nu_k)}{(1-KR_2(\nu_k))(1-KR_2(\eta(\nu_k)))}\right] > 1/K^2. \tag{2.3}$$

Then $C_0 = \emptyset$.

Proof. Suppose, to the contrary, that y is a solution of equation (1.1) belonging to the class C_0, say for $\nu \geq \nu_1$ for some sufficiently large $\nu_1 > \nu_0$. Integrating (1.1) from ν to l and then letting $l \to \infty$ gives

$$-y^\Delta(\nu) \geq \frac{1}{r(\nu)}\int_\nu^\infty q(x)f(y(\tau(x)))\Delta x.$$

Integrating the preceding inequality again from ν to ∞ and interchanging the order of integration (see [20, Lemma 2.1] or [21, Lemma 1]), we have

$$\begin{aligned} y(\nu) &\geq \int_\nu^\infty \frac{1}{r(s)}\int_s^\infty q(x)f(y(\tau(x)))\Delta x\Delta s = \int_\nu^\infty q(x)f(y(\tau(x)))\int_\nu^{\sigma(x)}\frac{1}{r(s)}\Delta s\Delta x \\ &\geq \int_\nu^\infty [R(\sigma(x)) - R(\nu)]q(x)f(y(\tau(x)))\Delta x, \end{aligned}$$

that is,

$$y(\nu) \geq \int_{\nu}^{\eta(\nu)} [R(\sigma(x)) - R(\nu)]q(x)f(y(\tau(x)))\Delta x + \int_{\eta(\nu)}^{\tau^{-1}(\nu)} [R(\sigma(x)) - R(\nu)]q(x)f(y(\tau(x)))\Delta x + \int_{\tau^{-1}(\nu)}^{\tau^{-1}(\eta(\nu))} [R(\sigma(x)) - R(\nu)]q(x)f(y(\tau(x)))\Delta x. \tag{2.4}$$

Since $\tau(\nu)$ is increasing, then it is not difficult to see that $\nu_k \in D_\tau$ implies $(\tau(\nu_k), \tau^{-1}(\eta(\nu_k)) \subset D_\tau$.

Since y is decreasing in C_0, letting $\nu = \nu_k$ in (2.4) yields

$$y(\nu_k) \geq Ky(\eta^{-1}(\nu_k))R_1(\nu_k) + Ky(\nu_k)R_2(\nu_k) + Ky(\eta(\nu_k))R_3(\nu_k),$$

that is,

$$y(\nu_k) \geq \frac{K}{1 - KR_2(\nu_k)} \left[y(\eta^{-1}(\nu_k))R_1(\nu_k) + y(\eta(\nu_k))R_3(\nu_k)\right]. \tag{2.5}$$

Similarly, setting $\nu = \eta^{-1}(\nu_k)$ in (2.4), we have

$$y(\eta^{-1}(\nu_k)) \geq \frac{K}{1 - KR_2(\eta^{-1}(\nu_k))} \left[y(\tau(\nu_k))R_1(\eta^{-1}(\nu_k)) + y(\nu_k)R_3(\eta^{-1}(\nu_k))\right], \tag{2.6}$$

and with $\nu = \eta(\nu_k)$, (2.4) implies

$$\begin{aligned} y(\eta(\nu_k)) &\geq \frac{K}{1 - KR_2(\eta(\nu_k))} \left[y(\nu_k)R_1(\eta(\nu_k)) + y(\tau^{-1}(\nu_k))R_3(\eta(\nu_k))\right] \\ &\geq \frac{KR_1(\eta(\nu_k))}{1 - KR_2(\eta(\nu_k))} y(\nu_k). \end{aligned} \tag{2.7}$$

Using (2.6) and (2.7) in (2.5), we obtain

$$\begin{aligned} y(\nu_k) \geq \frac{K}{1 - KR_2(\nu_k)} \Bigg[&\frac{KR_1(\nu_k)}{1 - KR_2(\eta^{-1}(\nu_k))} [y(\tau(\nu_k))R_1(\eta^{-1}(\nu_k)) + y(\nu_k)R_3(\eta^{-1}(\nu_k))] \\ &+ \frac{KR_3(\nu_k)}{1 - KR_2(\eta(\nu_k))} y(\nu_k)R_1(\eta(\nu_k)) \Bigg]. \end{aligned}$$

Since $y(\tau(\nu_k)) \geq y(\nu_k)$, we have

$$\begin{aligned} y(\nu_k) \geq K^2 y(\nu_k) \Bigg[&\frac{R_1(\nu_k)R_1(\eta^{-1}(\nu_k)) + R_1(\nu_k)R_3(\eta^{-1}(\nu_k))}{(1 - KR_2(\nu_k))(1 - KR_2(\eta^{-1}(\nu_k)))} \\ &+ \frac{R_3(\nu_k)R_1(\eta(\nu_k))}{(1 - KR_2(\nu_k))(1 - KR_2(\eta(\nu_k)))} \Bigg], \end{aligned}$$

which contradicts condition (2.3). Thus, $C_0 = \emptyset$, and this completes the proof of the lemma. $\square$

2.2. **Non-existence of C_2-type solutions.** Next, we establish conditions ensuring that the class C_2 is empty. Here we need to assume that there is a sequence

$$\{s_k\} \to \infty \text{ with } s_k \in A_\tau. \tag{2.8}$$

It is not difficult to see that $\eta(s_k) < s_k$.

Similar to what we did above, define:

$$R_4(s_k) = \int_{\tau^{-1}(\eta(s_k))}^{\tau^{-1}(s_k)} [R(s_k) - R(\sigma(x))]q(x)\Delta x,$$

$$R_5(s_k) = \int_{\tau^{-1}(s_k)}^{\eta(s_k)} [R(s_k) - R(\sigma(x))]q(x)\Delta x,$$

$$R_6(s_k) = \int_{\eta(s_k)}^{s_k} [R(s_k) - R(\sigma(x))]q(x)\Delta x.$$

Lemma 2.3. *Assume that there is a sequence* $\{s_k\}$ *satisfying* (2.2) *and a function* η *with* $\eta(\eta(s_k)) = \tau^{-1}(s_k)$ *such that*

$$\limsup_{k\to\infty} \left[\frac{R_6(s_k)R_4(\eta^{-1}(s_k)) + R_6(s_k)R_6(\eta^{-1}(s_k))}{(1 - KR_5(s_k))(1 - KR_5(\eta^{-1}(s_k)))} + \frac{R_4(s_k)R_6(\eta(s_k))}{(1 - KR_5(s_k))(1 - KR_5(\eta(s_k)))} \right] > 1/K^2. \tag{2.9}$$

Then $C_2 = \emptyset$.

Proof. Let $y \in C_2$ for $\nu \geq \nu_1$ and integrate (1.1) from ν_1 to ν to obtain

$$r(\nu)y^{\Delta}(\nu) \geq \int_{\nu_1}^{\nu} q(x)f(y(\tau(x)))\Delta x.$$

Integrating again from ν_1 to ν yields

$$y(\nu) \geq y(\nu_1) - y(\nu) \geq \int_{\nu_1}^{\nu} \frac{1}{r(u)} \int_{\nu_1}^{u} q(x)f(y(\tau(x)))\Delta x \Delta u$$

and interchanging the order of integration gives

$$y(\nu) \geq \int_{\nu_1}^{\nu} q(x)f(y(\tau(x))) \int_{\sigma(s)}^{\nu} \frac{1}{r(u)} \Delta u \Delta x \geq \int_{\nu_1}^{\nu} [R(\nu) - R(\sigma(x))]q(x)f(y(\tau(x)))\Delta x.$$

Therefore,

$$y(\nu) \geq K \int_{\tau^{-1}(\eta(\nu))}^{\tau^{-1}(\nu)} [R(\nu) - R(\sigma(x))]q(x)y(\tau(x))\Delta x \tag{2.10}$$

$$+ K \int_{\tau^{-1}(\nu)}^{\eta(\nu)} [R(\nu) - R(\sigma(x))]q(x)y(\tau(x))\Delta x$$

$$+ K \int_{\eta(\nu)}^{\nu} [R(\nu) - R(\sigma(x))]q(x)y(\tau(x))\Delta x. \tag{2.11}$$

Since $\tau(\nu)$ is increasing, we see that $s_k \in A_\tau$, and so $(\tau^{-1}(\eta(s_k)), s_k) \subset A_\tau$. Setting setting $\nu = s_k$ in (2.10) and using the fact that $y(\nu) \in C_2$ is increasing, we have

$$y(s_k) \geq KR_4(s_k)y(\eta(s_k)) + KR_5(s_k)y(s_k) + KR_6(s_k)y(\eta^{-1}(s_k)),$$

so that

$$y(s_k) \geq \frac{K}{(1 - KR_5(s_k))} \left[R_4(s_k)y(\eta(s_k)) + R_6(s_k)y(\eta^{-1}(s_k)) \right]. \tag{2.12}$$

Similarly, setting $\nu = \eta(s_k)$ in (2.10) gives

$$y(\eta(s_k)) \geq \frac{K}{(1 - KR_5(\eta(s_k)))} \left[R_4(\eta(s_k))y(\tau^{-1}(s_k)) + R_6(\eta(s_k))y(s_k) \right]$$

$$\geq \frac{KR_6(\eta(s_k))}{(1 - KR_5(\eta(s_k)))} y(s_k), \tag{2.13}$$

and with $\nu = \eta^{-1}(s_k)$ in (2.10),

$$y(\eta^{-1}(s_k)) \geq \frac{K}{(1 - KR_5(\eta^{-1}(s_k)))} \left[R_4(\eta^{-1}(s_k))y(s_k) + R_6(\eta^{-1}(s_k))y(\tau(s_k)) \right]. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12), we have

$$y(s_k) \geq \frac{K^2 y(s_k)}{(1 - KR_5(s_k))} \left[\frac{R_4(s_k)R_6(\eta(s_k))}{(1 - KR_5(\eta(s_k)))} \right.$$

$$+\frac{R_6(s_k)}{(1-KR_5(\eta^{-1}(s_k)))}\left[R_4(\eta^{-1}(s_k))+R_6(\eta^{-1}(s_k))\right]\bigg].$$

This contradicts (2.9), and so $C_2=\emptyset$. □

3. MAIN RESULTS AND EXAMPLES

By combining Lemmas 2.2 and 2.3 above, we obtain our main oscillation result in this paper.

Theorem 3.1. *Assume that there exist a function η satisfying* (2.1) *and two sequence $\{\nu_k\}$ and $\{s_k\}$ satisfying* (2.2) *and* (2.8)*, respectively. If conditions* (2.3) *and* (2.9) *hold, then every solution of* (1.1) *is oscillatory.*

The following examples illustrate this theorem.

Example 3.2. For $\mathbb{T}=\mathbb{R}$, consider the differential equation

$$y''(\nu)=\frac{q_0}{\nu^2}y(\nu(1+\cos\nu)) \quad \text{for } \nu>1, \tag{3.1}$$

where $q_0>0$ is a constant. Here, we have $q(\nu)=\frac{q_0}{\nu^2}$, $f(y)=y$, $\tau(\nu)=\nu(1+\cos\nu)$, and $\nu_0=1$.

Let $\nu_k=\frac{2\pi}{3}+2k\pi$ for $k\in\mathbb{N}$; then $\nu_k\in D_\tau$, $\tau(\nu_k)=\frac{\nu_k}{2}$, $\tau^{-1}(\nu_k)=2\nu_k$ and $\eta(\nu_k)=\sqrt{2}\nu_k$. Now $R(\nu)=\nu-1$ and $\sigma(\nu)=\nu$, so $R(\sigma(s))-R(\nu_k)=s-\nu_k$ and

$$R_1(\nu_k)=\int_{\nu_k}^{\eta(\nu_k)}(s-\nu_k)q(s)ds=q_0\int_{\nu_k}^{\sqrt{2}\nu_k}\left(\frac{1}{s}-\frac{\nu_k}{s^2}\right)ds=q_0\left(\ln\sqrt{2}+\frac{\sqrt{2}-2}{2}\right)=q_0c_1,$$

$$R_2(\nu_k)=\int_{\eta(\nu_k)}^{\tau^{-1}(\nu_k)}(s-\nu_k)q(s)ds=q_0\int_{\sqrt{2}\nu_k}^{2\nu_k}\left(\frac{1}{s}-\frac{\nu_k}{s^2}\right)ds=q_0\left(\ln\sqrt{2}+\frac{1-\sqrt{2}}{2}\right)=q_0c_2$$

and

$$\begin{aligned}R_3(\nu_k)=\int_{\tau^{-1}(\nu_k)}^{\tau^{-1}(\eta(\nu_k))}(s-\nu_k)q(s)ds&=q_0\int_{2\nu_k}^{2\sqrt{2}\nu_k}\left(\frac{1}{s}-\frac{\nu_k}{s^2}\right)ds\\&=q_0\left(\ln\sqrt{2}+\frac{1}{4}\left(\sqrt{2}-2\right)\right)=q_0c_3.\end{aligned}$$

From Eq. (2.3), we have

$$\frac{q_0^2c_1^2+q_0^2c_1c_3}{(1-q_0c_2)^2}+\frac{q_0^2c_1c_3}{(1-q_0c_2)^2}=\frac{q_0^2(c_1^2+2c_1c_3)}{(1-q_0c_2)^2}>1,$$

that is,

$$q_0^2(c_1^2+2c_1c_3)>(1-q_0c_2)^2=1-2q_0c_2+q_0^2c_2^2,$$

or,

$$q_0^2(c_1^2+2c_1c_3-c_2^2)>1-2q_0c_2.$$

Setting $c_4=c_1^2+2c_1c_3-c_2^2$, we have the quadratic inequality

$$q_0^2c_4+2q_0c_2>1.$$

Solving, we have

$$q_0\approx 3.36734.$$

Therefore, by Lemma 2.2, equation (3.1) has no C_0-type solutions for $q_0>3.36734$.

Let $s_k=2k\pi$ for $k\in\mathbb{N}$; then $s_k\in A_\tau$, $\tau(s_k)=2s_k$, $\tau^{-1}(s_k)=\frac{s_k}{2}$, and $\eta(s_k)=\frac{s_k}{\sqrt{2}}$. Now,

$$R_4(s_k)=\int_{\tau^{-1}(\eta(s_k))}^{\tau^{-1}(s_k)}(s_k-x)q(x)dx=q_0\int_{\frac{s_k}{2\sqrt{2}}}^{\frac{s_k}{2}}\left(\frac{s_k}{x^2}-\frac{1}{x}\right)dx$$

$$= q_0\left(2\sqrt{2}-2-\frac{\ln 2}{2}\right) = q_0 b_1,$$

$$R_5(s_k) = \int_{\tau^{-1}(s_k)}^{\eta(s_k)} (s_k - x)q(x)dx = q_0 \int_{\frac{s_k}{2}}^{\frac{s_k}{\sqrt{2}}} \left(\frac{s_k}{x^2}-\frac{1}{x}\right)dx$$
$$= q_0\left(2-\sqrt{2}-\frac{\ln 2}{2}\right) = q_0 b_2,$$

and

$$R_6(s_k) = \int_{\eta(s_k)}^{s_k} (s_k - x)q(x)dx = q_0 \int_{\frac{s_k}{\sqrt{2}}}^{s_k} \left(\frac{s_k}{x^2}-\frac{1}{x}\right)dx$$
$$= q_0\left(\sqrt{2}-1-\frac{\ln 2}{2}\right) = q_0 b_3.$$

From Eq. (2.9), we have

$$\frac{q_0b_1b_3+q_0b_1b_3}{(1-q_0b_2)(1-q_0b_2)} + \frac{q_0b_1b_3}{(1-q_0b_2)(1-q_0b_2)} = \frac{3q_0b_1b_3}{(1-q_0b_2)^2} > 1.$$

Proceeding as the last case, we have

$$q_0 \approx 2.2320.$$

Therefore, by Lemma 2.3, equation (3.1) has no C_2-type solutions for $q_0 > 2.2320$.

Thus, all solutions of equation (3.1) are oscillatory by Theorem 3.1 provided $q_0 > 3.36734$.

Example 3.3. For $\mathbb{T} = \mathbb{Z}$, consider the difference equations

$$\Delta^2 y(\nu) = \frac{q_0}{\nu^2} y(\tau(\nu)), \quad \nu > 1, \tag{3.2}$$

$q_0 > 0$ is a constant and

$$\tau(\nu) = \begin{cases} \nu - 2 & \text{if } \nu \text{ is even,} \\ \nu + 2 & \text{if } \nu \text{ is odd.} \end{cases}$$

Then,

$$\tau^{-1}(\nu) = \begin{cases} \nu + 2 & \text{if } \nu \text{ is even,} \\ \nu - 2 & \text{if } \nu \text{ is odd.} \end{cases}$$

If we take

$$\eta(\nu) = \begin{cases} \nu + 1 & \text{if } \nu \text{ is even,} \\ \nu - 1 & \text{if } \nu \text{ is odd,} \end{cases}$$

then $\eta(\eta(\nu)) = \tau^{-1}(\nu)$.

Let $\nu_k = 2k$, $k \in \mathbb{N}$; then $\nu_k \in D_\tau$, $\tau(\nu_k) = \nu_k - 2$, $\tau^{-1}(\nu_k) = \nu_k + 2$ and $\eta(\nu_k) = \nu_k + 1$. Now,

$$R_1(\nu_k) = \sum_{s=\nu_k}^{\eta(\nu_k)} (s+1-\nu_k)q(s) = \sum_{s=2k}^{2k+1} (s+1-2k)\frac{q_0}{s^2} = q_0\left[\frac{1}{(2k)^2}+\frac{1}{(2k+1)^2}\right],$$

$$R_2(\nu_k) = \sum_{s=\eta(\nu_k)}^{\tau^{-1}(\nu_k)} (s+1-\nu_k)q(s) = q_0\left[\frac{1}{(2k+1)^2}+\frac{1}{(2k+2)^2}\right],$$

and

$$R_3(\nu_k) = \sum_{s=\tau^{-1}(\nu_k)}^{\tau^{-1}(\eta(\nu_k))} (s+1-\nu_k)q(s) = q_0\left[\frac{1}{(2k+2)^2}+\frac{1}{(2k+3)^2}\right].$$

Therefore, by Lemma 2.2, equation (3.1) has no C_0-type solutions for $q_0 > 0.903125$.

Let $s_k = 2k+1$, $k \in \mathbb{N}$; then $s_k \in A_\tau$, $\tau(s_k) = s_k + 2$, $\tau^{-1}(s_k) = s_k - 2$ and $\eta(s_k) = s_k - 1$. We have

$$R_4(s_k) = \sum_{x=\tau^{-1}(\eta(s_k))}^{\tau^{-1}(s_k)} (s_k - x - 1))q(x) = q_0 \left[\frac{3}{(2k-3)^2} + \frac{2}{(2k-2)^2}\right].$$

$$R_5(s_k) = \sum_{x=\tau^{-1}(s_k)}^{\eta(s_k)} (s_k - x - 1)q(x) = q_0 \left[\frac{1}{(2k-1)^2} + \frac{0}{(2k)^2}\right].$$

and

$$R_6(s_k) = \sum_{x=\eta(s_k)}^{s_k} (s_k - x - 1)q(x) = q_0 \left[\frac{0}{(2k-1)^2} + \frac{-1}{(2k)^2}\right].$$

Therefore, by Lemma 2.3, equation (3.1) has no C_2-type solutions for $q_0 > 4.3$.

It follows from Theorem 3.1 that all solutions of (3.2) are oscillatory provided $q_0 > 4.3$.

4. **Concluding Remarks**

The results obtained here are the first ones for unstable type dynamic equations with mixed arguments and generalize many related results available in the literature. For example, if our time scale is the real numbers $\mathbb{T} = \mathbb{R}$, $r \equiv 1$, and $f(y) = y$, then we obtain the equation studied in [7]. Our approach employed the auxiliary function (2.1) in order to establish the oscillation of equation (1.1). Examples have been provided on different time scales to illustrate our results.

One direction for possible future research is to remove or modify condition (H_4) (which in a sense makes our equation linear or super-linear) to include sub-linear type equations.

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